A History of Site Valuation Rules: Functions and Empirical Evidence

Peter F. Colwell*
Tim F. Scheu**

Abstract. It is commonplace to think of the price of land as an amount per unit of area. This may be inappropriate, because it appears that the value of land increases at a decreasing rate as area increases in some situations, and frontage and depth may affect value differently. Various rules have been developed to aid in the process of estimating site value. This paper describes the functional forms of these rules and provides estimates of the parameters of these rules utilizing historical data. The hypotheses that value is a concave function of both frontage and depth cannot be rejected.

Introduction

Various rules have been developed by appraisers, judges and assessors to aid in the process of estimating site value. These site valuation rules can be divided into three categories: (1) depth rules; (2) frontage rules; and (3) area rules. A depth rule describes an increasing concave function (i.e., positive first and negative second derivatives) relating value to depth. Similarly, a frontage rule describes an increasing concave function relating value to frontage. Differences in these rules would imply that parcel shape matters. Alternatively if certain of these rules are identical for depth and frontage, then shape appears not to affect value and one is left with an area rule.

It is obligatory for anyone who would suggest that nonlinear site valuation rules are based in reality to explain how this could be possible. For these rules to reflect market prices, transaction costs must stand in the way of arbitrage eliminating nonlinear prices through further subdivision. This point has been clear since Colwell and Sirmans (1978, 1980) but has recently been restated by Brownstone and DeVany (1991).

The purpose of this paper is to trace the development of various site valuation rules, show how some of these rules utilize well-known functions, and to empirically investigate whether historical data support their early development and use. The second section of this paper traces the historical development of site valuation rules and shows that many of these rules, previously available only in tabular form, can be written in terms of their functional forms. Writing out the functions is a helpful step in empirically rejecting or not rejecting the rules. The third section of this paper describes the sales data from the New York City land market during the first half of the 1880s that are used in the fourth section. The fourth section develops the hedonic regression models, the hypotheses to be tested, and analyzes the results. The fifth and
final section offers some conclusions related to the relevance of the empirical work for the selection and application of site valuation rules.

Development of Site Valuation Rules

A site valuation rule relates site value to a measure of size. The simplest such rule is that value is proportional to parcel area or, in other words, that the value per unit of area is a constant. This simple rule has been rejected by sophisticated practitioners for at least a century. It is not surprising that the search for more reliable rules (i.e., more accurate predictors of sales prices) has continued throughout this period. While most interest has focused on depth as a measure of size, there has been some interest in nonlinear frontage effects and relatively recent interest in area rules.

Depth Rules

Various depth rules have been established to be used as guides in analyzing the declining marginal contribution of depth to value. The development and use of these depth rules have not proceeded without criticism. Critics claim that to apportion different values to different parts of a parcel of land, all of which is necessary as a unit for the production of income, would be speculative at best. Babcock wrote:

No great amount of inquiry into the fundamental concept of the depth table seems necessary to me. It seems to me that the value . . . is best discovered by an inquiry into its productivity rather than the depths, etc. (McMichael, 1935: 436).

Yet, depth may be a proxy for a parcel's productivity and productivity may diminish marginally with depth as suggested by depth rules. Whether depth is in fact a proxy for productivity and whether depth rules reflect reality are empirical questions that should not be dismissed out of hand.

The Development of Depth Rules

The development and proliferation of depth rules throughout the country can be traced to a basic need by assessors to use this type of tool where mass appraisal or the separation of value into land and building values is required. In the field of assessment, uniformity is important. During the period of 1900 to 1930 when depth tables were developing very rapidly, the assessment of property values for taxation purposes appeared to be rather crude and subject to corruption (Pollock and Scholz, 1926: 15). The use of depth rules by assessing authorities was promoted as a means by which more standardized assessments would result. By reducing the land assessment problem to one of referring to a table, depth rules simplified and standardized the assessment process.

Depth rules were generally developed by judges and assessors based on their
intuition. However, some rules may have been developed from rather extensive empirical evidence. Unfortunately there is insufficient documentation to precisely characterize this empirical evidence.

Although its origins are cloudy, the 4-3-2-1 Rule seems to be the first site valuation rule to gain acceptance (McMichael, 1951: 490). Utilizing a standard depth of 100 feet, it allocates 40% of the total value to the first 25% of standard depth, then 30%, 20% and finally 10% of the total value to each successive 25% of depth.

In 1866, Hoffman introduced the first depth rule to be recognized by the courts. The Hoffman Rule is based on the premise that the front half of a standard depth parcel of 100 feet is worth two-thirds of the total value (Zangerle, 1927: 108). This rule was used for many years in the New York City area. A number of strictly mathematical (i.e., unrelated to any empirical work) inconsistencies in the Hoffman Rule were later pointed out by Neill (Clar, 1936: 157). For example, at a depth of 25 feet, Neill computed two-thirds of the two-thirds of the value, allocated to 50 feet of depth, or 44% of the value. This contrasts with the Hoffman depth rule which allocated only 37.5% of total value to a depth of 25 feet. As real estate editor of the Evening Mail, a New York City newspaper, Neill published his table in 1904. This table became known as the Hoffman-Neill Rule. Neill's critique of Hoffman may have sensitized later rule developers to base their tables on mathematical functions.

Davies elevated the inquiry into site valuation rules to a new level and became the first to apply hedonics to real estate. He felt the use of either the Hoffman Rule, which was full of inconsistencies, or the Hoffman-Neill Rule, which he felt bore no relation to actual land sales, was foolish. In the 1891 edition of the *Diary of the Real Estate Board of New York*, Davies wrote how he developed his depth rule:

My first effort in the necessary direction was to obtain at great labor the average of the sales of ten thousand, two hundred parcels of varying depths, thus fixing numerous points, through which a curve could be run, and the resultant parabola stated in the formula, easy to remember, and moreover being based on actual sales, not subject to attack as having no foundation (Davies, 1912: 158).

The resultant rule, which became known as the Davies Rule, was the first depth rule not just based on mere intuition and anecdote, but based on actual market data.

Somers continued the tradition begun by Davies by investigating the contribution of varying depths on value in the St. Paul, Minnesota area. He also undertook an empirical investigation of market sales as a basis for his rule. After a "careful study" of over 2,000 parcels, he arrived at a tentative depth rule. He assigned 70% of the total value to the first 50 feet as is the case with the 4-3-2-1 Rule. After testing this curve in "other areas," he slightly modified his results. Somers, in constructing his depth rule, utilized both his empirical results and mathematical analysis. The result is a depth curve that follows a logarithmic function for the first 100 feet and then deviates from this function for depths greater than 100 feet. He later went to Cleveland, Ohio where he was hired by city officials to develop a similar rule for that city. This rule, known as the Cleveland Rule, assigns a percent of value up to depths of 700 feet. The Cleveland Rule and the Somers Rule have the same values for as far in depth as he had carried out the Somers Rule (McMichael, 1951: 505). If indeed
Exhibit 1
Definitions of Variables and Parameters

\[ A_c = \text{area of the comparable parcel,} \]
\[ A_s = \text{area of the subject parcel,} \]
\[ \alpha = \text{depth elasticity of value,} \]
\[ \beta = \text{frontage elasticity of value when different from one,} \]
\[ D_i = \text{the depth in feet of the } i^{th} \text{ parcel, and} \]
\[ D_s = \text{the depth in feet of the standard parcel,} \]
\[ F_i = \text{the number of front feet of the } i^{th} \text{ parcel,} \]
\[ p = \text{the price of the first square foot,} \]
\[ P = \text{price per front foot of the standard depth parcel,} \]
\[ V_c = \text{value (i.e., price) of the comparable property,} \]
\[ V_i = \text{value of the } i^{th} \text{ parcel,} \]
\[ \xi = \text{area elasticity of value.} \]

Source: Authors

Somers based his rules on empirical evidence, it is absolutely amazing that the evidence was precisely the same in St. Paul and "other areas" as in Cleveland.

Other rules were subsequently developed in other areas of the country. Many of these rules were developed for a particular land use type in a particular city or county. For example, a residential depth rule and a business depth rule were developed for use in Milwaukee, Wisconsin. A nearly complete list of depth rules can be found in McMichael (1951: 489–514).

**Depth Rule Functions**

Depth rules describe a concave function relating value to depth. Most depth rules shown in tabular form can be written as a mathematical function. Five different functional forms that fit a large number of depth rules are shown in Exhibit 2. Of the five functions, the Cobb-Douglas and logarithmic functions fit most of the depth rules. Of course, the parameters of these functions tend to differ across the various rules.

The extent of concavity varies across the different rules. This variation in concavity is illustrated in two ways. First, three indices of concavity are computed for each rule and shown in Exhibit 2. Second, the proportion of the value of a standard depth parcel at 25% of standard depth and at 75% of standard depth are computed and shown in Exhibit 2. These proportions would appear as percentages in depth rule tables.

Overall convexity is defined as the deviation of the function from a straight line between zero and the standard depth. The Overall Convexity Index (OCI) is as follows:

\[ OCI = 1 - \frac{2 \int_0^p V_i dD_i}{PF_i D_i} \]

where the variables are defined in Exhibit 1. The Overall Concavity Index functions
exactly like a Gini coefficient does in measuring overall inequality along a Lorenz curve. An Overall Concavity Index of zero indicates a linear relationship (i.e., no concavity), and greater concavity is indicated by a larger index.

The ratio of the negative second derivative to the first derivative is a concavity index that can be evaluated at any point on the function. That is,

$$CI(D_i) = \frac{-\frac{dV_i}{dD_i}}{\frac{dV_i}{dD_i}}$$

The second and third concavity indices in Exhibit 2 show the results for this measure evaluated at $D_i = .25D_i$ and $D_i = .75D_i$. Here again, zero indicates no concavity, and greater concavity is indicated by a larger index.

The second way the differences in convexity are highlighted is to take the ratio of the value function evaluation at a point to the value of the first foot of depth. That is,

$$\frac{V_i(D_i)}{PF_i}$$

This index is evaluated at $D_i = .25D_i$ and $D_i = .75D_i$. The results are shown in Exhibit 2. The extent to which $V_i(.25D_i)/PF_i$ deviates from .25 and $V_i(.75D_i)/PF_i$ deviates from .75 indicates the degree of concavity.

**Comparing Depth Rules**

The depth rules described in Exhibit 2 match the corresponding depth tables with great precision. The authors of the depth tables either used these very functions or borrowed from those who did. Some of the authors actually wrote out the functional form, others provided some hints (e.g., the two-thirds at 50 feet and two-thirds × two-thirds at 25 feet was a giveaway for the Hoffman-Neill Rule). The discovery of the function behind some of the rules required an educated guess (e.g., the 4-3-2-1 Rule is just another version of the sum-of-the-years-digits depreciation schedule).

The depth elasticity of value is indicated by $\alpha$ for each of the depth rules that utilize the Cobb-Douglas function (i.e., Function 1, Exhibit 2). For example, the Hoffman-Neill Rule has a depth elasticity of .585, whereas it is .5 for the Hobbs and Reeves Rules. The Milwaukee Rule falls in between with a depth elasticity of .51. Not surprisingly, the Overall Concavity Index exceeds .32 for all the rules of this type except for the Hoffman-Neill Rule. That is, the Hoffman-Neill Rule shows the least concavity of rules of this type.

A glance at the concavity indices evaluated at one quarter and three quarters of standard depth reveals that the greater concavity occurs “early” in the Cobb-Douglas function. This is true of most, but not all, of the depth rule types as will become clear.


### Exhibit 2

**Depth Rule Functional Forms**

<table>
<thead>
<tr>
<th>Rule</th>
<th>$V_i = PF_i(D_i/D_s)\alpha$</th>
<th>$D_s$</th>
<th>$OCl$</th>
<th>$CI(.25D_s)$</th>
<th>$CI(.75D_s)$</th>
<th>$V_i(.25D_s)$</th>
<th>$V_i(.75D_s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cobb-Douglas</td>
<td>$\alpha$</td>
<td>.500</td>
<td>125</td>
<td>.3333</td>
<td>.0200</td>
<td>.0067</td>
<td>.5000</td>
</tr>
<tr>
<td></td>
<td></td>
<td>.585</td>
<td>100</td>
<td>.2616</td>
<td>.0166</td>
<td>.0055</td>
<td>.4444</td>
</tr>
<tr>
<td></td>
<td></td>
<td>.510</td>
<td>100</td>
<td>.3241</td>
<td>.0196</td>
<td>.0065</td>
<td>.4931</td>
</tr>
<tr>
<td></td>
<td></td>
<td>.500</td>
<td>100</td>
<td>.3329</td>
<td>.0200</td>
<td>.0067</td>
<td>.5000</td>
</tr>
<tr>
<td>2. Logarithmic: $V_i = PF_i(\log(1 + 9D_i/D_s))^\pi$</td>
<td>$\pi$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Cleveland</td>
<td>1.07</td>
<td>100</td>
<td>.3256</td>
<td>.0261</td>
<td>.0112</td>
<td>.4884</td>
</tr>
<tr>
<td></td>
<td>III. Apartment</td>
<td>1.14</td>
<td>135</td>
<td>.2989</td>
<td>.0244</td>
<td>.0108</td>
<td>.4661</td>
</tr>
<tr>
<td></td>
<td>III. Residential</td>
<td>1.25</td>
<td>120</td>
<td>.2692</td>
<td>.0218</td>
<td>.0102</td>
<td>.4330</td>
</tr>
<tr>
<td></td>
<td>Leenhouts</td>
<td>1.71</td>
<td>120</td>
<td>.1188</td>
<td>.0110</td>
<td>.0075</td>
<td>.3182</td>
</tr>
<tr>
<td></td>
<td>McMahon</td>
<td>1.36</td>
<td>100</td>
<td>.2222</td>
<td>.0192</td>
<td>.0096</td>
<td>.4022</td>
</tr>
<tr>
<td></td>
<td>Montreal</td>
<td>1.11</td>
<td>100</td>
<td>.3102</td>
<td>.0251</td>
<td>.0110</td>
<td>.4755</td>
</tr>
<tr>
<td></td>
<td>Somers</td>
<td>1.07</td>
<td>100</td>
<td>.3256</td>
<td>.0261</td>
<td>.0112</td>
<td>.4884</td>
</tr>
<tr>
<td></td>
<td>Stafford</td>
<td>1.07</td>
<td>100</td>
<td>.3256</td>
<td>.0261</td>
<td>.0112</td>
<td>.4884</td>
</tr>
<tr>
<td>3. Parabolic: $V_i = PF_i(\psi + \tau D_i/D_s - \theta(D_i/D_s)^2)$</td>
<td>$\psi$</td>
<td>$\tau$</td>
<td>$\theta$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>4-3-2-1</td>
<td>0.0</td>
<td>1.8</td>
<td>.8</td>
<td>100</td>
<td>.2666</td>
<td>.0114</td>
</tr>
<tr>
<td></td>
<td>M-C</td>
<td>0.1</td>
<td>1.0</td>
<td>.1</td>
<td>100</td>
<td>.1333</td>
<td>.0021</td>
</tr>
<tr>
<td>4. Square Root: $V_i = PF_i((\varphi(D_i/D_s)^5 - \rho)$</td>
<td>$\sigma$</td>
<td>$\varphi$</td>
<td>$\rho$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Davies</td>
<td>.0352</td>
<td>1.45</td>
<td>.226</td>
<td>100</td>
<td>.2284</td>
<td>.0175</td>
</tr>
<tr>
<td>5. Jerret: $V_i = PF_i(2D_i/(D_i/D_s)$</td>
<td></td>
<td></td>
<td></td>
<td>100</td>
<td>.2274</td>
<td>.0160</td>
<td>.0114</td>
</tr>
</tbody>
</table>

*Source: Authors*

The final two indices show that the variation among the Cobb-Douglas depth rules at three quarters of standard depth is relatively trivial compared to the variation at one quarter of the standard depth.

A larger family of depth rules with greater variation in concavity employs a logarithmic function as shown in Exhibit 2 for Function 2. The Leenhouts Rule is nearly linear with $\pi = 1.71$ and the OCI = .1188 ($\pi = 2.21$ yields an OCI of approximately zero). The Cleveland-Stafford-Somers Rule is the most concave of this type with the OCI close to the maximum OCI found among the Cobb-Douglas type rules.
Again the concavity is greater "early" in the function, but the differences between concavity at one quarter and three quarters of standard depth are less for the "Logarithmic" rules than those found for the Cobb-Douglas rules. The final two indices show that although the variation in the "Logarithmic" rules is much greater at three quarters of standard depth than that of the Cobb-Douglas depth rules, this variation is still relatively small compared to the variation among the "Logarithmic" rules at one quarter of the standard depth.

Both the Martin-Chicago (M-C) and 4-3-2-1 Rules are parabolas as shown by Function 3 in Exhibit 2. The Martin-Chicago Rule reaches its maximum at 500 feet of depth, well beyond most standard depths and certainly beyond its own 100 foot standard depth. The 4-3-2-1 Rule reaches its maximum at 112.5 feet, however the function is only thought to apply up to a standard depth of 100 feet. Beyond 100 feet of depth, the 4-3-2-1 Rule does not appear to follow any function. The vertical intercept for the Martin-Chicago Rule does not go through the origin, but curiously allocates 10% of value to parcels with no depth. Still, the Overall Concavity of the Martin-Chicago Rule is very small, second only to the Leenhouts Rule. The curiosity is that these parabolic rules display more concavity "late" in the function than "early" in the function. This is in contrast with all the other depth rules. The 4-3-2-1 Rule is shown by the last index to deviate more from the linear at three quarters of standard depth than any other rule.

The Davies and Jerret Rules are each unique. While one can appreciate the appeal of the computational simplicity of the Jerret Rule in an age prior to calculators and computers, one wonders how Davies was able to concoct such a convoluted form let alone estimate its parameters as he indicated he was able to do. These rules are very similar to each other in overall concavity and in their concavity at the first quarter of standard depth. As with all but the parabolic rules, the concavity declines from the first quarter to that found at the third quarter of standard depth. However, this difference is practically nothing for the Jerret Rule.

**Frontage Rules**

While many depth rules were developed and used by assessors and appraisers, little is found in the literature that explicitly addresses frontage rules. Implicitly, the developers of depth rules assumed that value is proportional to frontage, ceteris paribus. Ross, a Los Angeles appraiser, developed a frontage rule for which he devised a table to show the effect of frontage on value while holding depth constant (McMichael, 1951: 498). The Ross Rule for residential parcels uses standard depths of 125 to 150 feet and categorizes residential parcels into four different classifications. In general, for all classes of residential property, increasing frontage results in marginally declining value. It is a curiosity that according to the Ross Rule, at some point for each of the classifications, increasing frontage actually results in absolutely declining value.

**Area Rules**

There is a rather large empirical literature that indicates there may be area rules
(i.e., a concave relationship between land value and the area of the parcel; see Asabere, Nov. 1981, Aug. 1981; Asabere and Colwell, 1984, 1985; Asabere and Huffman, Summer 1991, Spring 1991; Brownstone and DeVany, 1991; Colwell and Sirmans, 1980). In the appraisal literature, Dilmore has developed several area rules to assist in adjusting for disparities in size between a subject property and a comparable property (Dilmore, May–June 1976, May 1976, 1971). The development of these area rules was also intended to increase the number of comparable properties (i.e., by providing the means for making adjustments) and reduce the problem of inconsistencies arising from adjustments for size.

The area rules are based on the premise that increases in area cause decreases in unit price. After testing several types of curves, Dilmore found the “learning curve” is best suited for land prices. The learning curve is a Cobb-Douglas function and its parameter is an elasticity coefficient.

Dilmore defines the ratio of comparable to subject unit prices as follows:

$$\frac{A_{c}^{\xi}/A_{i}^{\xi}}{A_{c}/A_{i}} = \frac{V_{c}/A_{c}}{V_{i}/A_{i}},$$

where variables and parameters are defined in Exhibit 1. Simplifying equation (4) and solving for $V_{i}$ yields:

$$V_{i} = \frac{V_{c}}{A_{c}^{\xi}} A_{i}^{\xi},$$

where $V_{c}/A_{c}^{\xi}$ equals an estimate of the price of the first square foot, based upon the sale of a comparable parcel. All Dilmore has contributed is an expression for the value of the first square foot in lieu of estimation via regression.

For rectangular parcels, Dilmore’s approach can be interpreted as nothing more than having a depth rule and a frontage rule of the Cobb-Douglas type, where the frontage and depth elasticities are identical as follows:

$$V_{i} = \frac{V_{c}}{F_{c}^{\xi}D_{i}^{\xi}} F_{i}^{\xi}D_{i}^{\xi}.$$

If $\xi$ equals one in equations (4), (5) and (6), value is proportional to area, depth and frontage. This implies that prices are linear and it is a useful construct to think of the value per square foot. Another way to think about this is that constant returns in parcel area implies increasing returns to scale in frontage and depth. That is, doubling both frontage and depth quadruples value if $\xi$ equals one, but doubling both also quadruples area.

If $\xi$ is between zero and one in equations (4), (5) and (6), value increases at a decreasing rate with increases in area, depth, or frontage. This means that prices are nonlinear and value per square foot is not a particularly useful construct. It also means that the commonly used value per front foot is also not a particularly useful construct.
Combining Frontage and Depth Rules

There is no theoretical reason for frontage and depth elasticities to be equal. If anything, there is a theoretical reason for the frontage elasticity to exceed the depth elasticity (Colwell and Cannaday, 1990; Colwell and Scheu, 1989). If these elasticities differ, then the frontage rule differs from the depth rule as in equation (7).

\[ V_i = pF_i^\beta D_i^\alpha, \]  

where \( \beta \neq \alpha \).

If \( \beta \) is between zero and one in equation (7), then frontage too has a diminishing marginal impact on value. This assumption has been utilized in the theory of Colwell and Scheu (1989) and Colwell and Cannaday (1990). The empirical justification for this view is to be found in Kowalski and Colwell (1986) and, to a lesser extent, in Colwell and Scheu (1989).

If \( \beta \) equals one as in the various rules specified in Exhibit 2, then the price of frontage is linear. This would imply that it makes sense to discuss the price per front foot (as is frequently done for commercial or for lakefront property) assuming that depth is constant.

Data

The data are analyzed to determine whether the early development and use of depth rules are empirically justifiable. The data consist of vacant land sales in New York City during the period 1800 to 1885 (Real Estate Record Association, 1880–1884). The use of this historical data is particularly significant in that the early development and use of depth rules took place in New York at approximately this time (i.e., with Hoffman earlier and Neill later).

The data consist of 187 vacant land sales from a sample area of the New York market. Each of the observations front on avenues. The sales data for parcels that front on streets were omitted from the sample because nearly all the observations were either at the standard depth or at twice the standard depth (i.e., through-block parcels) and thus do not exhibit sufficient variation in depth to be informative about depth rules. In New York City, streets run east and west and avenues run north and south. The New York avenue data sample area is bounded by First Avenue and Eleventh Avenue on the east and west, respectively, and 50th Street and 120th Street on the south and on the north, respectively. In addition to the selling price of each parcel, other variables were recorded. These include the frontage and depth, the date of sale, and the parcel’s address. A location variable was computed from the address and historical accounts of the location of peak value.

The location of a parcel of real estate is an important factor to consider in estimating the parcel’s value. Some major location factors that were present in New York City for the period 1880 through 1884 are distance to the central business district, proximity to the intersection of 50th Street and 5th Avenue, proximity to Central Park and distance from either the East River or the Hudson River.

During the study period, the area north of 59th Street was the outskirts of town. Further development depended upon the growth of the mass transit system. It was
also during this time that Madison Avenue and Fifth Avenue north of 50th Street started to become fashionable. In August 1879, Vanderbilt purchased property between 51st and 52nd Streets adjacent to Fifth Avenue (Real Estate Record Association, 1898: 81). His purchases were imitated by a number of his friends and other investors. This sector continued north to the park. For many years there was a reluctance to develop adjacent to the park. Its marshy ground had been used as a dumping site for raw sewage, a practice that continued even after it officially became a park. Central Park was also inhabited by the homeless.

During the sample period, proximity to either the East River or Hudson River also had a negative effect on value. There are several reasons for this influence that can be gathered from the historical accounts. The first reason was that with very little commercial activity taking place in the area there was no incentive to develop the waterfront for shipping and receiving goods. Another reason for the negative influence on value was the crime rate. The waterways were the most accessible and fastest transportation arteries and enhanced the opportunity for criminal getaways. The final factor for this negative influence was the climate. It was colder and more damp along the waterfront than it was inland. The technology for heating homes was still fairly primitive, so in order to find a less harsh climate, people preferred to live away from the water. Based on the historical record, the intersections of 50th Street and 5th Avenue appears to be the point where local land values were highest. If the point of maximum value were farther south, it would not matter much for the creation of a location variable because no observations are included below 50th Street.

One of the more profound data questions is whether there is sufficient variation in the parcel dimensions, especially depth, found in the sample. There appears to be no such problem with the avenue data, as can be seen in Exhibit 3. Nevertheless, one wonders whether the apparent variation in depth is merely the result of a very few extreme observations. In fact, 70 out of 187 observations have depths other than 100 feet. Thus, we believe that reliable estimates of the influence of depth are obtainable.

Regression Models, Hypotheses and Results

Models

All of the variables and hypotheses developed in the prior sections are brought together in Exhibit 4. These models differ only with respect to the depth rule or depth rule family represented. In Models I and II, the critical depth parameters \( \alpha \) and \( \pi \) are
### Exhibit 4
The Empirical Models

<table>
<thead>
<tr>
<th>Model Number</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$\ln V_i = \lambda + \beta \ln F_i + \pi \ln (D_i/D_f) + \delta U_i + \omega T$</td>
</tr>
<tr>
<td>II</td>
<td>$\ln V_i = \lambda + \beta \ln F_i + \pi \ln (\text{LOG}) + \delta U_i + \omega T$</td>
</tr>
<tr>
<td>III</td>
<td>$\ln V_i = \lambda + \beta \ln F_i + \zeta(\ln(4\cdot3\cdot2\cdot1)) + \delta U_i + \omega T_i$</td>
</tr>
<tr>
<td>IV</td>
<td>$\ln V_i = \lambda + \beta \ln F_i + \zeta \ln (M\cdot C) + \delta U_i + \omega T_i$</td>
</tr>
<tr>
<td>V</td>
<td>$\ln V_i = \lambda + \beta \ln F_i + \zeta \ln (DAVIES) + \delta U_i + \omega T_i$</td>
</tr>
<tr>
<td>VI</td>
<td>$\ln V_i = \lambda + \beta \ln F_i + \zeta \ln (JERRET) + \delta U_i + \omega T_i$</td>
</tr>
</tbody>
</table>

where
- $\ln V_i$ = the natural log of the selling price of parcel $i$ in dollars,
- $\ln F_i$ = the natural log of the frontage of parcel $i$ in feet,
- $\ln (D_i/D_f)$ = the natural log of the ratio of the depth of parcel $i$ to the depth of a standard parcel,
- $U_i$ = the distance in thousands of feet from 50th Street and 5th Avenue,
- $T_i$ = the time in months from January of 1880 to the sale of parcel $i$,
- $\ln \text{(LOG)}$ = the natural log of the logarithmic type of depth rule excluding the parameter $\pi$ (LOG(1 + 9D/D_f))
- $\ln (4\cdot3\cdot2\cdot1)$ = the natural log of the 4-3-2-1 depth rule (with a linear approximation beyond 100 feet of depth), (1.8D/D_f - 0.8(D/D_f)^2) for D_f, < 100 and (1 + .0032(D - 100)) for D_f, > 100
- $\ln (M\cdot C)$ = the natural log of the Martin-Chicago depth rule, (1 + 1D/D_f - 1(D/D_f)^2),
- $\ln \text{DAVIES}$ = the natural log of the Davies depth rule, ((1.45(0.0352 + D/D_f)^5 - 0.226), and
- $\ln \text{JERRET}$ = the natural log of the Jerret depth rule, (2D/(D + D_f)).

**Source:** Authors

to be estimated directly. In Models III through VI, a depth parameter, $\zeta$, is to be estimated that bends the rule if it deviates from one. If $\zeta$ is greater than one, it indicates that, in reality, the depth relation is less concave than the rule, whereas it is more concave than the rule if $\zeta$ is less than one. Model I represents the Cobb-Douglas family of depth rules. The depth rules in Models (II) through (VI) are named explicitly. The frontage rule in all models is of the Cobb-Douglas type. The location variable was included in all models. This variable was computed as the linear distance between the point of hypothesized maximum value and the parcel. Finally, the time variable (i.e., month of sale) was used in all models.

**Hypotheses**

The purpose of the regression estimates is to provide for tests of the following hypotheses:

- We expect a coefficient on the log of frontage between zero and one, indicating a concave frontage rule;
- we expect a coefficient on the log of the depth ratio in Model 1 between zero and one indicating a concave depth rule;
we expect that a coefficient on the log of the depth ratio in Model 1 is between .5 and .585, the range of elasticity found in the Cobb-Douglas depth rules;

- we expect that the coefficient on the log of depth is less than that on the frontage in Model 1 indicating that area rules are not justified because frontage elasticity is greater than depth elasticity and that the cited theory and empirical work is confirmed;

- we expect the coefficient on the depth rule variable in Model II to fall between 1.07 and 1.71, the range found in the "Logarithmic" depth rules;

- we expect the coefficient on the depth rule variable in Models III through VI to equal 1.0 indicating the concavity in the rule is found in the data;

- we expect the coefficient on the location variable to be negative indicating that values fall with distance from the point selected as the peak;

- we expect the coefficient on the time variable to be positive indicating that values typically were increasing during the sample period.

There can be no nested test (e.g., Box-Cox and Box-Tidwell) to identify the functional forms that produce log likelihood estimates insignificantly different from that which maximizes the log-likelihood function. The reason is that all the functional forms considered do not belong to a single family of functions in the sense that each can be considered a special case of a more general function.

**Results**

The regression results support the notion of concave site valuation rules (i.e., diminishing marginal contributions of frontage and depth). Exhibit 5 shows the results of the regression analysis. All of the coefficients are significantly different from zero at the 95% level of confidence. The t-ratio for each estimated coefficient is shown in parentheses. While the results indicate that there is a concave depth rule, there is a question of whether there is a concave frontage rule. Furthermore, it is not possible to reject the hypothesis that there is a concave area rule.

The coefficient on depth is significantly less than one in Model I. Note that this is properly a one-tailed test. The indication is that it is not possible to reject the hypothesis that there is a depth rule like those that were developed about the same time as the study period. While this coefficient is slightly outside the range found among the Cobb-Douglas rules, it is almost precisely the magnitude of the implied coefficient in the Hoffman-Neill Rule. If a question remains about the robustness of this result, because it is thought that depths seldom deviate from the standard depth, it is possible to re-estimate the model with all standard depth parcels omitted. Even though depth appears quite variable (see Exhibit 3), only seventy observations are at depths other than 100 feet. Re-estimating Model I using only these seventy observations yields

\[
\ln V_i = 6.824 + .8451 \ln F_i + .567 \ln D_i/D_j - .083 U_i + .012 T_i.
\]

The depth elasticity coefficient is virtually unchanged by this procedure. It is peculiar that the rate of appreciation is doubled, but the other coefficients are also virtually
unchanged. It is safe to say that the data are sufficiently variable to reveal the influence of depth on value.

In Model II, the coefficient on the depth variable is between the extremes found in the depth rules using the Logarithmic functional form. It is, however, not significantly different than either extreme magnitude (i.e., 1.07 and 1.71). Also, it is not significantly different than the magnitude that produces zero overall net concavity (i.e., 2.21). Of course, this result raises the question of whether the value-depth function is indeed concave; however it should be recognized that the test of a difference from 2.21 is not strictly a test of whether the function is linear (i.e., zero overall net concavity cannot be equated to linearity if the function can be part convex and part concave).

Looking at the results for the 4-3-2-1 and Martin-Chicago Rules (i.e., Models III and IV) gives the impression that the true model is more concave than the Martin-Chicago Rule and less concave than the 4-3-2-1 Rule. That is, the coefficient for 4-3-2-1 is substantially greater than one and the coefficient for Martin-Chicago is substantially below one. However, these coefficients are not significantly different from one in either case. Thus, it is not possible to state that the true model differs from either of these rules.

The coefficient on the Davies Rule (Model V) is almost exactly one, indicating that the concavity of the Davies Rule matches that found in the sample data almost exactly. Finally, the coefficient on the Jerret Rule (Model VI) is close to one and certainly not significantly different from one. Thus, it is impossible to reject the hypothesis that the Jerret Rule is correct.

The coefficient on frontage is significantly less than one in Models I through VI. This indicates that it is possible to reject the hypothesis that there is a linear frontage rule. It should be remembered that the depth rule creators, with only one exception, believed that value is proportional to frontage. This finding to the contrary should be viewed as a deviation from the received doctrine. However, the magnitude of the deviation is not large.

Despite the fact that the frontage and depth elasticity estimates in Model I are of the approximate hypothesized absolute and relative magnitudes, the difference between them is not statistically significant. One way to produce this test is to substitute lot area for the depth ratio variable. The result of this re-estimation of the model is as follows:

\[ \ln V_i = 4.528 + 0.1861 \ln F_i + 0.589 \ln Ai - 0.087 U_i + 0.006 T_i . \]

\[ \text{(.730)} \quad \text{(2.559)} \quad \text{(-8.801)} \quad \text{(2.571)} \]

The test is then whether the coefficient on the frontage variable is significantly different from zero if the alternative hypothesis is that the frontage elasticity is different than the depth elasticity, or significantly greater than zero if the alternative hypothesis is that frontage elasticity is greater than depth elasticity. The result shows that it is not possible to reject the hypothesis that there is an area rule. That is, both alternative hypotheses are rejected.

It was hypothesized that, north of 50th Street, parcel values decline with distance from the intersection of 50th Street and 5th Avenue, holding other things constant. We find that value decreases 8.7% per 1,000 feet of distance. This is the right order of magnitude for the study period. Mills found that unit values decline by 49% and 33% per mile for Chicago in 1857 and 1873, respectively (1971).
Exhibit 5
Regression Results

<table>
<thead>
<tr>
<th>Model</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
<th>V</th>
<th>VI</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R^2$</td>
<td>.505</td>
<td>.504</td>
<td>.505</td>
<td>.505</td>
<td>.505</td>
<td>.503</td>
</tr>
<tr>
<td>Variables</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\ln F_r$</td>
<td>.775</td>
<td>.778</td>
<td>.777</td>
<td>.776</td>
<td>.776</td>
<td>.781</td>
</tr>
<tr>
<td>$\ln (O/D)_{t-1}$</td>
<td>.589</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2.559)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\ln (LOG)_{t-1}$</td>
<td></td>
<td>1.545</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2.468)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\ln (4.3 \cdot 2 \cdot 1)_{t-1}$</td>
<td>1.405</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2.509)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\ln M-C_{t-1}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2.525)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\ln (DAVIES)_{t-1}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1.002</td>
<td></td>
</tr>
<tr>
<td>(2.538)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\ln (JERRET)_{t-1}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1.228</td>
</tr>
<tr>
<td>(2.375)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$U_{t-1}$</td>
<td>-.087</td>
<td>-.087</td>
<td>-.087</td>
<td>-.087</td>
<td>-.087</td>
<td>-.087</td>
</tr>
<tr>
<td>(-8.801)</td>
<td>(-8.788)</td>
<td>(-8.794)</td>
<td>(-8.796)</td>
<td>(-8.798)</td>
<td>(-8.801)</td>
<td></td>
</tr>
<tr>
<td>$t_{t-1}$</td>
<td>.006</td>
<td>.006</td>
<td>.006</td>
<td>.006</td>
<td>.006</td>
<td>.006</td>
</tr>
<tr>
<td>(2.571)</td>
<td>(2.574)</td>
<td>(2.579)</td>
<td>(2.575)</td>
<td>(2.572)</td>
<td>(2.571)</td>
<td></td>
</tr>
<tr>
<td>Constant</td>
<td>7.242</td>
<td>7.233</td>
<td>7.232</td>
<td>7.239</td>
<td>7.233</td>
<td>7.241</td>
</tr>
</tbody>
</table>

*Source: Authors*

Based on the historical record, it was hypothesized that parcel values were increasing during the sample period. The empirical results support this hypothesis. The coefficient on the time of sale variable is significantly different from zero (and, of course, significantly positive in all models). The coefficient indicates that parcel value was growing at a compound rate of .6% per month.

**Conclusions**

The site valuation rules analyzed in this paper are divided into three categories: depth rules, frontage rules, and area rules. The functional form can be found for a large number of depth rules. These forms are specified where possible. Frontage rules, on the other hand, are generally neglected in the literature. We assume that frontage rules, if they exist, take the form of a Cobb-Douglas function. Similarly, area rules are assumed to be Cobb-Douglas functions in which the frontage and depth elasticities are identical. The use of historical data is shown to be significant in that the early development and use of depth rules took place in New York City, the place of our study, at approximately the time of our study.
The results provide support for the development and use of depth rules. We are neither able to reject any of the depth rules nor are we able to distinguish among them in any meaningful way. That is, we cannot rank the rules. We cannot even identify the best rule. Of course, the rules are quite similar. That is, their differences represent a range of concavity, none are linear and none are convex. The magnitudes of the coefficient estimates suggest that the degree of concavity found in the market falls within the range found in the various rules.

The results also provide support for the use of frontage rules. That is, we cannot reject the hypothesis that there is a concave frontage rule in the market. Unfortunately perhaps, the developers of depth rules generally assumed that the frontage relation is linear. Thus, our results are in conflict with depth rule doctrine.

We cannot say that the depth elasticity differs significantly from the frontage rule. Therefore, it is not possible to reject the hypothesis that there is an area rule (i.e., shape does not matter). The estimated area rule suggests that value is a concave function of parcel area.

The next logical step would be to test the relevance of these site valuation rules for contemporary usage. Kowalski and Colwell (1986) and Colwell and Schau (1989) provide the first steps in this direction by considering the relevance of the Cobb-Douglas frontage and depth rules for valuing land. Kowalski and Colwell's results show a pronounced concave depth rule as well as a concave frontage rule on data from the Detroit area between 1975 and 1983. Colwell and Schau's results show pronounced concave depth rules but possibly linear frontage rules in Champaign-Urbana, Illinois between 1970 and 1974 as well as in Bloomington-Normal, Illinois between 1975 and 1982. The similarities of the magnitudes of frontage and depth elasticities across these studies, and between these studies and the present study, suggest that site valuation rules have relevance across time and space. However, more empirical work is needed to truly demonstrate the stability of these relationships.

References


SUMMER 1994

This research was funded, in part, by grants from the Office of Real Estate Research at the University of Illinois, Urbana-Champaign.