Acoustic cavitation series: part two

Bubble phenomena in sound fields: part one

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This paper presents a mainly theoretical review of the physical aspects of the behaviour of bubbles in sound fields. Firstly, an equation for the radial motion, including the effects of liquid compressibility, is critically presented. The equilibrium radius and its stability are then considered, followed by a presentation of results concerning the small-amplitude radial oscillations of gas and vapour bubbles.

KEYWORDS: ultrasonics, cavitation, bubble dynamics

Introduction

The large compressibility of gases compared with that of liquids gives rise to a very strong interaction between bubbles and oscillating pressure fields. This is an essential aspect of acoustic cavitation and this article is intended as a review of the physical aspects of these processes with an emphasis on principles and results rather than derivations. However, derivations are included when they are straightforward or add to the physical understanding of the discussion. Since other articles in this series will deal with nucleation and mass diffusion this paper assumes that fully developed bubbles already exist in the liquid and investigates their behaviour when immersed in a sound field. A good part of the discussion is concerned with spherical bubbles, since this mode of oscillation is the one most strongly coupled with the sound field and most readily understood theoretically. Large-amplitude oscillations and non-spherical motion will be considered in part two of this paper, which also contains a discussion of one of the important cooperative effects of bubbles in liquids, namely the speed of sound in bubbly mixtures.

Reasons of space have prevented us from including other topics such as cryogenic liquids (see however, the section on vapour bubbles), sonoluminescence, and chemical effects. For these omissions the reader is referred to other articles in this series and to reviews available in the literature. The books edited by Rozenberg and the two recent conference proceedings edited by Lauterborn and van Wijngaarden also contain much valuable material.

Spherical dynamics in a compressible liquid

Among the recent developments in the understanding of bubble dynamics is the realization of the importance of the liquid compressibility through the high-speed holo-cinematographic techniques of Lauterborn and his group. This has sparked off a renewed interest in the mathematical modelling of these effects, which after the first studies received relatively little attention for a long time. Many of the available equations are in fact equivalent as they incorporate only first-order, acoustic corrections to the incompressible theory. We begin by discussing the basis of these equations, which will serve to derive an equation of motion of fundamental importance in the following.

We consider a single spherical bubble isolated in a liquid that extends to infinity. The starting points are the equations of conservation of mass and momentum for the liquid which we write as

\[
\frac{1}{\rho} \left( \frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial r} \right) + \nabla \cdot \mathbf{u} = 0,
\]

(1)

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} = - \frac{1}{\rho} \frac{\partial p}{\partial r},
\]

(2)

Here \( u = u e_r \) is the velocity vector entirely directed along the radial direction, \( \rho \) and \( p \) denote density and pressure, and \( r \) is the distance measured from the centre of the spherical bubble. With the assumption of isentropic motion in the liquid we may write...
\[ \rho \mathbf{v} = c^{-2} \rho \mathbf{v} \cdot \mathbf{v}, \quad \rho \mathbf{h} = \rho^{-1} \mathbf{v}, \]  
(3)

where \( h \) is the enthalpy and \( c \) is the speed of sound. Furthermore, since a purely radial motion is irrotational, we may also introduce a velocity potential \( \phi \) such that \( \mathbf{u} = \nabla \phi \). In terms of these quantities the preceding equations may be written

\[ \nabla^2 \phi + \frac{1}{c^2} \left( \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial r} \right) = 0 \]  
(4)

\[ \frac{\partial \phi}{\partial t} + \frac{1}{2} (\nabla \phi)^2 + h = 0 \]  
(5)

The second equation was obtained by integrating (2) and the constant of integration set to zero, thus implicitly assuming that \( \phi \) tends to zero at large distances from the bubble and that the enthalpy is referred to its value at infinity. These equations can be simplified if the speed of sound in the liquid is assumed to be large. Indeed we note that, by a Taylor series expansion,

\[ \frac{\partial \phi}{\partial t} + \frac{1}{2} (\nabla \phi)^2 + h = 0 \]  
(6)

and similarly,

\[ \frac{\partial \phi}{\partial t} + \frac{1}{2} (\nabla \phi)^2 + h = 0 \]  
(7)

where \( c_\infty \) are the static values of the speed of sound, pressure, and density at large distances from the bubble. Correct to order \( c^{-2} \) the preceding equations can therefore be written as

\[ \nabla^2 \phi + \frac{1}{c^2} \left( \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial r} \right) = 0 \]  
(8)

\[ \frac{\partial \phi}{\partial t} + \frac{1}{2} (\nabla \phi)^2 + h = 0 \]  
(9)

We can now consider the limiting form of these equations near and far from the bubble. In the former limit, the finite speed of propagation of the signals is unimportant and the liquid behaves as if it were essentially incompressible. Thus the appropriate form for the near-field is

\[ \nabla^2 \phi = 0, \]  
(10)

\[ \frac{\partial \phi}{\partial t} + \frac{1}{2} (\nabla \phi)^2 + h = 0 \]  
(11)

that is, the customary incompressible formulation. Far from the bubble, on the other hand, one may expect

\[ \nabla^2 \phi + \frac{1}{\rho \omega c_\infty^2} \frac{\partial \phi}{\partial t} = 0 \]  
(12)

\[ \frac{\partial \phi}{\partial t} + \frac{\rho - \rho_\infty}{\rho_\infty} = 0 \]  
(13)

from which we obtain

\[ \nabla^2 \phi - \frac{1}{c_\infty^2} \frac{\partial^2 \phi}{\partial t^2} = 0 \]  
(14)

An approximate model, valid simultaneously in the near and far-field, can be obtained by using the wave equation (14) together with the incompressible Bernoulli integral (9). Indeed, (14) is correct in the far-field, and its difference from (10) in the near-field is negligible. Similarly, (9) differs from (13) by the presence of the term \( (\nabla \phi)^2 \), which is small in the far-field. This mathematical formulation was first proposed by Keller and Kolodner, and subsequently resumed and generalized by Epstein and Keller and Keller and Miksis. It may be shown that it is equivalent in essence to the first steps of a singular-perturbation type of approach which can be extended to yield higher-order terms of a formal series expansion in the Mach number. Using then (9) and (14) and following the same procedure used by Keller in the papers referred to, the following equation of motion for the radial oscillations of a bubble of instantaneous radius \( R(t) \) is obtained

\[ \left( 1 - \frac{\dot{R}}{c} \right) R \ddot{R} + \frac{3}{2} R \dot{R}^2 \left( 1 - \frac{\dot{R}}{3c} \right) = \right. \]
\[ \left. \left( 1 + \frac{\dot{R}}{c} \right) \frac{p(R, t) - p(R + R/c)}{p} + \frac{R}{pc} \frac{dp(R, t)}{dt} \right) \]  
(15)

Here \( p(R, t) \) is the static pressure plus the pressure of the sound field driving the oscillations of the bubble. Dots denote differentiation with respect to time, and the subscript \( \infty \) has been dropped from \( p \) and \( c \). This equation differs slightly from that of Keller and Miksis since the terms introduced by the viscosity of the liquid have been dropped in view of their smallness for liquids of interest in acoustic cavitation. The liquid pressure just outside the bubble, \( p(R, t) \), is connected to the pressure \( p(R, t) \) acting on the inner side of the bubble surface by the condition on the normal stresses

\[ p(R, t) = p(R, t) + 2a/R + \rho \dot{R}/R \]  
(16)

where \( a \) is the surface tension coefficient and \( \mu \) the liquid viscosity. (The inconsistency in dropping the viscous term in (15) and retaining it here is only formal since in the former case this term is always negligible, whereas in this relation it can have an important effect for small bubbles.) In the incompressible limit \( c \to \infty \), and (15) becomes the well-known equation

\[ \ddot{R} + \frac{3}{2} \dot{R}^2 = \rho^{-1} [p(R, t) - p_s(t) - 2a/R - 4\mu \dot{R}/R] \]  
(17)
Derived from a synthesis of the incompressible and acoustic approximations, (15) is accurate only up to terms of first order in \( c^{-1} \). Therefore several equivalent forms of the equation can be obtained by using (17) to express in a different way the terms of order \( c^{-1} \). For instance, from (17),

\[
\frac{R}{c} \left( \frac{p(R, t) - p_s}{\rho} \right) = \frac{R}{c} \left( \frac{3}{2} R \frac{\partial R}{\partial t} + \frac{3}{2} R^2 \right) + O(c^{-3})
\]

so that, upon insertion into (15), we find

\[
\left( 1 - \frac{R}{c} \right) R \frac{\partial R}{\partial t} + \frac{3}{2} \left( 1 - \frac{R}{3c} \right) R^2 = \left( 1 + \frac{R}{c} \right) H
\]

which is the form given by Herring. The question as to which of (15) and (18) is 'closer to the truth' can be understood in two different ways. Which form would be obtained by approximating to order \( c^{-3} \) an equation of higher accuracy? A moment's thought shows that the answer can only be found by comparison with an equation correct to all orders, and hence cannot be obtained at present. There is however a more useful sense in which to look at this question, namely, which equation will give the least error when used for moderate values of \( R/c \)? This matter could be examined numerically, for example by following the lines of the study by Hickling and Plesset, but such a comparison has not been made. *A priori* one can say that the form (15) is presumably better because the coefficients of \( R/c \) are smaller than those in (18). Thus, (18) shows the term \( R \frac{\partial R}{\partial t} \) to give an unphysical contribution of opposite sign to \( R \) when \( R/c > 1/2 \), whereas the same happens in (15) only for \( R/c > 1 \).

Another equation that accounts for compressibility effects is due to Gilmore and is based on the Kirkwood-Bethe approximation

\[
\left( 1 - \frac{R}{c} \right) R \frac{\partial R}{\partial t} + \left( 1 - \frac{R}{c} \right) R \frac{dH}{dt} = \frac{R}{c} \frac{dR}{dt} \left( 1 + \frac{R}{c} \right) H
\]

Here \( H \) is the value of the enthalpy at the bubble wall and \( C \) is the speed of sound in the liquid evaluated from the liquid equation of state rather than held constant as in (15) and (18). The starting point for the Kirkwood-Bethe approximation is the fact that the velocity potential of an outgoing acoustic wave has, by (14), the form \( \phi = f(r/c) \), so that \( r \frac{\partial \phi}{\partial r} = f'(r/c) \). Equation (5) then shows that \( \phi = \frac{1}{4} u_0^2 \) also propagates with the outward velocity \( c_m \). To go beyond the acoustic approximation one may assume that the quantity \( \phi + \frac{1}{2} u_0^2 \) propagates outward with the characteristic speed given by the sum of the fluid and pressure perturbation velocities, \( \phi + c \). This is similar to what happens in the case of one-dimensional waves. The mathematical form of this assumption is

\[
\frac{\partial}{\partial \mathbf{r}} \left[ r \left( \phi + \frac{1}{2} u_0^2 \right) \right] + (\phi + c) \frac{\partial}{\partial \mathbf{r}} \left[ r \left( \phi + \frac{1}{2} u_0^2 \right) \right] = 0
\]

At the bubble wall the convective derivative \( \partial/\partial \mathbf{r} + u \partial/\partial \mathbf{r} \) is the total time derivative indicated by dots in (19), and the terms \( \partial \phi/\partial \mathbf{r}, \phi \partial \phi/\partial \mathbf{r} \) can be expressed by means of this derivative using (2) and (1), respectively. With these substitutions the preceding equation evaluated at the bubble wall becomes (19). An analogous, but not completely equivalent, approximation can be obtained if \( \phi \) is substituted for \( h + \frac{1}{2} u_0^2 \) in (20). This procedure, known as the non-linearization technique, is also in use.

If in (19) \( H \) is substituted by \( u \mathbf{r}^2 \), as in (6), \( C \) is held constant, and the term in \( C^2 \) is dropped, then (15) is recovered. As it stands, by comparison with available equations known to be correct to second order, it can be said that it is not a consistent second-order approximation in a formal sense. However (19) has been tested by Hickling and Plesset against a full numerical solution and has been found to be very accurate up to Mach numbers greater than unity for the case of a collapsing cavity in a static pressure field. This confirmation however does not imply that the equation can be applied directly to the situation of interest in acoustic cavitation in which the motion is driven by an ambient pressure field. This is because, in its derivation, explicit use is made of the fact that no disturbances propagate along the ingoing characteristic lines. That is, that no pressure waves travel towards the bubble. In this more general situation the equation lacks both a theoretical basis and numerical support. The second-order equation of Tomita and Shimada also applies only to the case of a constant ambient pressure.

The relative merits of the different equations cannot be assessed experimentally as yet because, as shown in Ref. 18, they all give essentially identical results at low Mach number, and deviate markedly only for \( R/c \) close to unity, a situation in which data are very difficult to obtain.

### Equilibrium radius

In static conditions the pressure in the liquid is uniform, \( p_l = p_m \), a constant, and the radius of the bubble \( R \) must satisfy (16) in the form

\[
p_l(R) = p_m + 2a/R
\]

If the mass of gas contained in the bubble is \( m_G \) and the liquid vapour pressure at the liquid temperature \( T_m \) is \( p_v \), then at a good approximation

\[
p_l(R) = \frac{3m_G}{4\pi M} \frac{\rho_G}{T_m} R - \frac{p_v}{R^3}
\]

where \( M \) is the molecular weight of the gas and \( \rho_G \) the universal gas constant. Substitution into (21) then gives an explicit equation for the equilibrium radius. The nature of the stability of this equilibrium can be determined by formal means by using the dynamic equations of the previous section, but it is simpler to proceed intuitively observing that the surface tension force in (21) tends to collapse the bubble whereas the gas pressure tends to expand it. Consider then a small increase of the bubble radius. The condition for stability is evidently that, at the new radius, the surface tension force be greater than the gas pressure, which requires the derivative of \( 2a/R \) to be greater than that

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of \( p_v(R) \). This condition is always satisfied if \( p_\infty > p_v \), whereas in the opposite case it requires that \( R < R_{ct} \), where the critical value \( R_{ct} \) is given by

\[
R_{ct} = \frac{4}{3} \frac{\sigma}{p_v - p_\infty}
\]

At 20°C the vapour pressure of water is 0.023 bar, at 50°C it is 0.12 bar and at 90°C it is 0.69 bar. Hence it is clear that, for most cases of truly static conditions, the equilibrium will be stable. However, the preceding result can also be approximately applied to situations in which the pressure varies slowly with respect to the bubble resonance period (see the following section) so that inertial effects are negligible. In this case \( R \) must be understood as the instantaneous value of the radius at the pressure \( p_w \), which may be connected to the initial value \( R_i \) at the static ambient pressure \( p_\infty \), by the isothermal relation

\[
\left( p_v - p_\infty + \frac{2\sigma}{R_i} \right) R_i^3 = \left( p_v - p_\infty + \frac{2\sigma}{R} \right) R^3
\]

which follows from (21) and (22). Setting \( R = R_i \), we find then the following minimum value of the pressure which will maintain the stability of a bubble of undisturbed radius \( R_i \):

\[
p_v - p_\infty, B = \frac{4}{3} \frac{\sigma}{R_i} \left( \frac{2}{p_v - p_\infty} \right) R_i^3
\]

This limiting value is known as the Blake threshold.\(^{3,6,7,8}\) For water at 20°C and \( p_\infty - p_v = 1 \) bar we find

\[
p_v - p_\infty, B = 6.6 \times 10^{-4} \text{ bar}
\]

values of the pressure amplitude above which the limiting value is known as the Blake threshold.\(^{3,6,7,8}\) This limiting value is known as the Blake threshold.\(^{3,6,7,8}\) The growth process triggered by a decrease of the pressure below the Blake threshold must of course be analysed by means of the complete dynamical equation of motion (15) or of its incompressible approximation (17).

Numerical calculations show that, for sound frequencies much below the resonance frequency of the bubble, (24) is meaningful in the sense that it yields values of the pressure amplitude above which the bubble radius grows by one or more orders of magnitude during one pressure cycle.\(^{41-44}\) Note that the Blake threshold predicts only a rapid expansion of an existing bubble nucleus. It says nothing about any implosion.

At higher frequencies dynamic effects are significant and a simple expression for a threshold for substantial growth is not readily obtained.\(^{45}\) However, numerical solutions do show that a sort of threshold can be identified, in the sense that a slight increase in the sound pressure gives rise to large expansions and very violent collapses. This difference in behaviour reflects the difference between stable and transient acoustic cavitation that is observed when the liquid is subjected to an acoustic pressure. In the stable regime bubbles are seen to oscillate more or less stably over many cycles, whereas at higher pressure amplitude, in transient cavitation, a large expansion is followed by a violent collapse during which the bubble often disintegrates. The acoustic emissions of the two cavitation regimes are quite different.

During stable cavitation the average radius may vary in time due to a gain or loss of the gas contained in the bubble. This process occurs spontaneously in a supersaturated or undersaturated liquid, but it can also be substantially affected by the sound field itself, in which case one speaks of rectified diffusion. These topics will be addressed in another article in this series (see also Refs 46-48). It is important to note, however, that these processes occur very slowly compared to the period of the sound fields of practical interest, so that the corresponding variation in \( R \) can be effectively uncoupled from the oscillatory motion.

Finally we wish to anticipate that in the case of bubbles containing mostly vapour one may have a situation of dynamic equilibrium in the sense that, in a sound field, the bubble will oscillate about an average radius, but it will collapse very fast as soon as the pressure wave is turned off. We shall consider this question at greater length in the section on vapour bubbles.

### Small amplitude oscillations of gas bubbles

Many of the fundamental concepts and quantities of interest in the discussion of bubble behaviour in sound fields arise in the context of small amplitude oscillations under the action of a periodic pressure perturbation, which we write as

\[
p(R, t) = p_\infty (1 - e^{i\omega t})
\]

where \( |\epsilon| \ll 1 \). In the case of gas bubbles these oscillations occur about an equilibrium radius \( R_0 \) satisfying (21) and we therefore set

\[
R(t) = R_0 [1 + X(t)]
\]

where \( |X| \) will be assumed to be of order \( \epsilon \). The pressure perturbation in the bubble is proportional to the surface displacement and therefore we write

\[
p_0(R, t) = p_0(R_0) (1 - \phi X)
\]

Upon substitution of (25)–(27) into (15) and linearization we find

\[
-\rho R_0^2 \omega^2 [1 - i \omega R_0/c] + 4\mu \omega \phi + \phi \rho (R_0) - 2\sigma/\omega R_0 X
\]

\[
= (1 - i \omega R_0/c) e^{i\omega t}
\]

Steady state oscillations will occur at the frequency \( \omega \). We can therefore use relations such as \( \omega^2 X = -X \) and similar ones to cast this equation in the form of a driven harmonic oscillator

\[
\ddot{X} + 2\beta \dot{X} + \omega_0^2 X = \alpha e^{i\omega t}
\]

where

\[
\begin{align*}
2\beta &= \frac{4\mu}{\rho R_0^2} + \frac{\omega_0^2 R_0}{c} + \frac{\rho \phi (R_0)}{\rho \omega R_0^2} \text{Im } \\
\omega_0^2 &= \frac{p_0(R_0)}{\rho R_0^2} \left[ \text{Re } \phi - \frac{2\sigma}{\rho R_0 p_0(R_0)} \right] \\
\alpha &= \frac{p_\infty}{\rho R_0^2} \left( 1 - \frac{\omega R_0}{c} \right)
\end{align*}
\]

This form is interesting because it enables one to
identify at a glance the different contributions to the effective damping constant $\beta$ and to the effective resonance frequency $\omega_0$. We use the adjective 'effective' because, unlike the case of a true mechanical oscillator, these quantities are found to depend on $\omega$ both explicitly and through $\phi$. Equation (29) shows energy dissipation to arise from viscosity, sound radiation, and thermal and mass diffusion effects, the latter two accounted for by $\text{Im} \phi$. Similarly, the restoring force is provided by surface tension and internal pressure variations.

Explicit results require the evaluation of the quantity $\phi$. The simplest assumption is that the gas in the bubble behaves polytropically so that

$$p_i(R, t) = p_i(R_0, t) \left[ \frac{R}{R_0} \right]^{3\kappa},$$

(32)

where $\kappa$ is the polytropic exponent. Upon comparison with (27) we then find $\phi = 3\kappa$ so that there is no effect on the damping, while the resonance frequency becomes

$$\omega_0 = \frac{p_i(R_0)}{\rho R_0^2} \left[ 3\kappa - \frac{2\sigma}{R_0 p_i(R_0)} \right]$$

(33)

or, using (21)

$$\omega_0 = \frac{p_i}{\rho R_0^2} \left[ 3\kappa + (3\kappa - 1) \frac{2\sigma}{R_0 p_i} \right]$$

(34)

These results are essentially due to Minnaert. 

A priori one can expect that $\kappa \approx 1$ (that is, isothermal behaviour) at low frequency or, more precisely, when the thermal diffusion length $\chi_G/(2\omega)^{1/2}$ is large compared to the radius. (Here $\chi_G$ is the thermal diffusivity of the gas defined in terms of the thermal conductivity $K_G$, the density $\rho_G$, and the constant pressure specific heat $C_{P_G}$ by $\chi_G = K_G/\rho_G C_{P_G}$.) Conversely the gas should behave approximately adiabatically (that is, $\kappa \approx \gamma$, where $\gamma$ is the ratio of the specific heats) when $\chi_G/(2\omega)^{1/2} \ll R_0$. As (32) shows, the very use of a polytropic exponent implies that the gas pressure in the bubble is uniform and depends only on the instantaneous value of the radius. Both of these circumstances are of course approximately realized only in the adiabatic and isothermal limit and therefore the question of the appropriate value of $\kappa$ in the intermediate region is, from a fundamental point of view, meaningless. However, we may continue to use this concept to characterize in an average way the restoring force and the energy exchange of the bubble using the following definition suggested by a comparison of (30) and (33)

$$\kappa = \frac{1}{3} \text{Re} \phi$$

(35)

The determination of $\phi$ requires the solution of the conservation equations of mass, momentum, and energy in the bubble. The procedure is straightforward but tedious and we refer the reader to the original papers for a derivation of the results discussed here for the case in which mass transfer across the bubble interface is neglected. For our purpose it is sufficient to present an approximation to $\phi$ which is valid essentially with the only restriction that $(k_G R_0)^3 \ll 1$, where $k_G$ is the wave-number in the gas.

$$\phi = \frac{3\gamma \theta^2}{\theta \left[ \theta + 3(\gamma - 1)A_+ \right] - 3(\gamma - 1)(\theta A_+ - 2)^2}$$

(36)

where

$$A_+ = \frac{\sin \theta \pm \sin \theta}{\cosh \theta - \cos \theta},$$

(37)

and $\theta$ is the ratio of the bubble radius to the gas thermal diffusion length

$$\theta = R_0 (2\omega/\chi_G)^{1/2}$$

(38)

With this result we can obtain the following explicit expression for the polytropic exponent

$$\kappa = \frac{\gamma \theta^3}{\theta^2 \left[ \theta + 3(\gamma - 1)A_+ \right]^2 + 9(\gamma - 1)^2 (\theta A_+ - 2)^2}$$

(39)

and for the contribution $\beta_\text{th} = \frac{1}{2} \text{Im} \phi$ to the damping constant arising from $\phi$,

$$\beta_\text{th} = \frac{9\gamma(\gamma - 1) p_i(R_0)}{\rho \chi_G} \times \frac{\theta A_+ - 2}{\theta^2 \left[ \theta + 3(\gamma - 1)A_+ \right]^2 + 9(\gamma - 1)^2 (\theta A_+ - 2)^2}$$

(40)

This expression accounts for the energy losses of thermal origin. In general, and particularly at low ambient pressure, energy losses due to mass diffusion in and out of the bubble also exist. The analysis in this case is rather involved and can be found in Refs 55-59. For $\theta \to 0$, $\kappa \to 1$ and

$$\beta_\text{th} \to \frac{\gamma - 1}{10\gamma} \frac{p_i(R_0)}{\rho \chi_G}$$

(41)

whereas for $\theta \to \infty$, $\kappa \to \gamma$ and

$$\beta_\text{th} \to \gamma(\gamma - 1) p_i(R_0) \left( \frac{\chi_G}{2\omega \gamma} \right)^{1/2}$$

(42)

In the first case (low frequency) we see from (29) that viscous damping dominates at small radii, and thermal damping at larger radii, while the acoustic contribution is negligible. In the second case thermal damping is unimportant and energy is dissipated by viscosity and acoustic radiation.

We show in Figs 1 and 2 graphs of $\kappa$ and of $\beta_\text{th}$ as a function of $\theta$. The graphs are for bubbles of monatomic ($\gamma = 5/3$) and diatomic ($\gamma = 7/5$) gases as a function of $\theta$. The ratio of the bubble radius to the gas thermal diffusion length in the gas...
Fig. 2 Normalized thermal damping constant, $\beta_{th} = \frac{[\rho X(\gamma/R_0)] \beta_{th}}{\beta_{th}}$, according to (40) for bubbles of monatomic ($\gamma = 5/3$) and diatomic ($\gamma = 7/5$) gases as a function of $\theta = \frac{R_0}{(2k^2/\rho G)}$, the ratio of the bubble radius to the thermal diffusion length in the gas. The horizontal portions of the lines are the approximation (41) valid for small $\theta$ and the dashed lines are the approximation (42) valid for large $\theta$.

Given by (39) we can compute $\omega_0$ from (34). This relation is shown in Fig. 7 again for the same air-water case. The dashed lines indicate the isothermal and adiabatic values.

The previous theoretical results for $\kappa$ have recently received excellent experimental support. Essentially the experiment comprises the measurement of the radius of a bubble and its position, in a known non-uniform pressure field. As will be explained in part two the position of the oscillating bubble is extremely sensitive to the value of $\omega_0$ so that the theory can be put to a

Fig. 3 Total damping constant for air bubbles in water according to (29) and (40) for $R_0 = 0.1$ cm as a function of the driving sound frequency (continuous line) and its thermal and acoustic contributions (dashed lines). The viscous contribution has the value 1 s$^{-1}$ and is out of scale. The open circle marks the position of the resonance and the dashed horizontal segment in the upper left is the low frequency approximation (41).

$\beta_{th} = \frac{[\rho X(\gamma/R_0)] \beta_{th}}{\beta_{th}}$ as functions of $\theta$ for monatomic ($\gamma = 5/3$) and diatomic ($\gamma = 7/5$) gases. The behaviour of the total damping constant $\beta$ as a function of the frequency $\nu = \omega/2\pi$ is shown in Figs 3-6 for the case of water and air at 20°C and 1 bar ($\sigma = 72.8$ erg cm$^{-2}$, $\rho = 1$ g cm$^{-3}$, $X_G = 0.20$ cm$^{-2}$ s, $\gamma = 1.4$) for different values of the equilibrium radius. These figures also show the individual components of the total damping according to (29). Finally, with $\kappa$

Fig. 4 Total damping constant for air bubbles in water according to (29) and (40) for $R_0 = 10^{-2}$ cm as a function of the driving sound frequency (continuous line) and its thermal and acoustic contributions (dashed lines). The viscous contribution has the value 200 s$^{-1}$ and is out of scale. The open circle marks the position of the resonance and the dashed horizontal segment on the left is the low frequency approximation (41).

Fig. 5 Total damping constant for air bubbles in water according to (29) and (40) for $R_0 = 10^{-2}$ cm as a function of the driving sound frequency (continuous line) and its thermal, acoustic, and viscous contributions (dashed lines). The open circle marks the position of the resonance.

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Experimental data for the damping constant can be found in Refs 60-65. Agreement between theory and experiment is also good, but the scatter in the data is somewhat greater. New, more sophisticated experiments to measure this quantity would be desirable.

The preceding theoretical developments concern a bubble far from any boundary. The effect of boundaries has been studied analytically and experimentally $^{66-71}$. It is found that a rigid boundary decreases the resonance frequency, whereas a free boundary increases it.

**Vapour bubbles**

A bubble containing pure vapour collapses in a static pressure field under the action of pressure and surface tension forces and there is no static equilibrium radius in this case. However, the bubble can be stabilized by a sound field inducing volume pulsations. To see the physical mechanism by which a positive stiffness can result in this way, we consider the extreme case in which the temperature changes in the bubble are determined primarily by evaporation and condensation. Thus, we neglect the energy required to heat and cool the bubble contents and to perform the expansion work.

![Image](https://example.com/image.png)
Let us imagine a slight increase $\Delta R$ in the radius. The mass of liquid $\Delta m$ which vaporizes to keep the bubble full is of the order of $\Delta m = 4\pi R^2 \rho, \Delta R$, where $\rho$ is the vapour density, and the associated energy requirement is $L \Delta m$, where $L$ is the latent heat. If the thermal diffusion length in the liquid, $(\chi/2\omega)^{1/2}$, is small compared with $R$, this thermal energy causes a cooling of the bubble surface of the order of $4\pi R^2 (\chi/2\omega)^{1/2} \rho C_1 \Delta T = L \Delta m$, where $C_1$ is the liquid specific heat. Following this cooling the pressure in the bubble falls by $\Delta p \approx \partial p/\partial T \Delta T$, where $\partial p/\partial T$ is the slope of the saturation line. The same mechanism clearly leads to a pressure increase if the radius is decreased by $\Delta R$. Hence a restoring force of the elastic type, $F = -k \Delta R$, arises with $k$ obtained from the previous argument in the form

\[ k = 4\pi R^2 (\omega/\lambda L)^{1/2} \left( L \rho_v L \rho_L C_1 \right) \partial p_v/\partial T \tag{43} \]

To obtain an expression for the resonance frequency we need only divide $k$ by the equivalent mass $M = 4\pi R^2 \rho$ of the oscillating bubble according to the general formula $\omega^2 = k/M$. The result is

\[ \omega^2 R^2 = \frac{1}{\chi L} \left[ \frac{\rho_v}{\rho_L C_1 \rho_\lambda L} \frac{\partial p_v}{\partial T} \right]^2 \tag{44} \]

This expression is valid provided that the pressure drop caused by the phase change dominates over the other contributions to $\Delta p$. The most important of these, $\Delta p'$, is due to the expansion itself and can be estimated by means of the isothermal relation $R \rho_v = \text{constant}$. The condition $\Delta p \gg \Delta p'$ gives

\[ \frac{3p_v}{R} < \left( \frac{\omega}{\lambda L} \right)^{1/2} \frac{\rho_v}{\rho_L C_1 \rho_\lambda L} \frac{\partial p_v}{\partial T} \tag{45} \]

When this condition is not satisfied the stiffness of the bubble (if present at all) arises from the same mechanism at work in the case of a gas bubble. It is obvious from the preceding arguments that $\Delta p$ is significant only if the frequency is so large that a non-negligible temperature difference exists between the bubble surface and the rest of the liquid. Since this temperature difference depends on the mass that evaporates and condenses, and since this quantity, being proportional to $\rho_v$, is a strongly increasing function of temperature, the minimum frequency for a positive stiffness will be a decreasing function of temperature. For instance, for a vapour bubble in water with $R = 0.1$ cm, one finds a positive stiffness threshold at $\nu \approx 640, 1,4, 0.022, 0.011$ s$^{-1}$ for $T = 20, 50, 90, 100^\circ C$ respectively.

A rather interesting peculiarity of the vapour bubble case is the presence of a second resonance frequency at low frequency. The mechanism causing this behaviour can be understood by a mechanical example. Consider an oscillator subject to both a stabilizing force, $f_1 = -k_1 X$, and a destabilizing force, $f_2 = k_2 X$, and let $k_1$ and $k_2$ be functions of $\omega$. If $B$ is the damping constant and $M$ the mass, the displacement per unit exciting force at the frequency $\omega$ is given by

\[ \left\{ -M \omega^2 + 2B \omega + k_1(\omega) - k_2(\omega) \right\} \]

If, as $\omega \to 0$, $k_1$ and $k_2$ decrease slower than $\omega^2$ this quantity can become large when $k_1 \approx k_2$. This condition defines then a second resonance in addition to the ordinary one given by $\omega^2 = [k_1(\omega) - k_2(\omega)]/M$.

References:
