2. What are the Euler formulas? By what very important idea did we obtain them?
3. How did we proceed from 2π-periodic to general-periodic functions?
4. Can a discontinuous function have a Fourier series? A Taylor series? Why are such functions of interest to the engineer?
5. What do you know about convergence of a Fourier series? About the Gibbs phenomenon?
6. What do you know about convergence of a Fourier cosine series? About the Gibbs phenomenon?
7. What do you know about convergence of a Fourier sine series? About the Gibbs phenomenon?
8. What is the Fourier transform? The discrete Fourier transform?
10. What are Sturm–Liouville problems? By what idea are they related to Fourier series?

**FOURIER SERIES.** In Probs. 11, 13, 16, 20 find the Fourier series of \( f(x) \) as given over one period and sketch \( f(x) \) and partial sums. In Probs. 12, 14, 15, 17–19 give answers, with reasons. Show your work detail.

11. \( f(x) = \begin{cases} 0 & \text{if } -2 < x < 0 \\ 2 & \text{if } 0 < x < 2 \end{cases} \)
12. Why does the series in Prob. 11 have no cosine terms?
13. \( f(x) = \begin{cases} 0 & \text{if } -1 < x < 0 \\ x & \text{if } 0 < x < 1 \end{cases} \)
14. What function does the series of the cosine terms in Prob. 13 represent? The series of the sine terms?
15. What function do the series of the cosine terms and the series of the sine terms in the Fourier series of \( e^x (-5 < x < 5) \) represent?
16. \( f(x) = |x| (-\pi < x < \pi) \)
17. Find a Fourier series from which you can conclude that \( -1/3 + 1/5 - 1/7 + \cdots = \pi/4 \).
18. What function and series do you obtain in Prob. 16 by (termwise) differentiation?
19. Find the half-range expansions of \( f(x) = x (0 < x < 1) \).
20. \( f(x) = 3x^2 (-\pi < x < \pi) \)

**GENERAL SOLUTION**

Solve, \( y'' + \omega^2 y = r(t) \), where \( |\omega| \neq 0, 1, 2, \cdots, r(t) \) is 2π-periodic and

21. \( r(t) = 3t^2 (-\pi < t < \pi) \)
22. \( r(t) = |t| (-\pi < t < \pi) \)

**MINIMUM SQUARE ERROR**

23. Compute the minimum square error for \( f(x) = x/\pi (-\pi < x < \pi) \) and trigonometric polynomials of degree \( N = 1, \cdots, 5 \).
24. How does the minimum square error change if you multiply \( f(x) \) by a constant \( k? \)
25. Same task as in Prob. 23, for \( f(x) = |x|/\pi (-\pi < x < \pi) \). Why is \( E^N \) now much smaller (by a factor 100, approximately)?

**FOURIER INTEGRALS AND TRANSFORMS**

Sketch the given function and represent it as indicated. If you have a CAS, graph approximate curves obtained by replacing \( \infty \) with finite limits; also look for Gibbs phenomena.

26. \( f(x) = x + 1 \) if \( 0 < x < 1 \) and 0 otherwise; by the Fourier sine transform
27. \( f(x) = x \) if \( 0 < x < 1 \) and 0 otherwise; by the Fourier integral
28. \( f(x) = a \) if \( a < x < b \) and 0 otherwise; by the Fourier transform
29. \( f(x) = x \) if \( 0 < x < a \) and 0 otherwise; by the Fourier cosine transform
30. \( f(x) = e^{-2x} \) if \( x > 0 \) and 0 otherwise; by the Fourier transform
Fourier series concern periodic functions \( f(x) \) of period \( p = 2L \), that is, by definition \( f(x + p) = f(x) \) for all \( x \) and some fixed \( p > 0 \); thus, \( f(x + np) = f(x) \) for any integer \( n \). These series are of the form

\[
f(x) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)
\]

(Sec. 11.2)

with coefficients, called the Fourier coefficients of \( f(x) \), given by the Euler formulas (Sec. 11.2)

\[
a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) \, dx, \quad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx
\]

\[
b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx
\]

where \( n = 1, 2, \ldots \). For period \( 2\pi \) we simply have (Sec. 11.1)

\[
f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)
\]

(1*)

with the Fourier coefficients of \( f(x) \) (Sec. 11.1)

\[
a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx.
\]

Fourier series are fundamental in connection with periodic phenomena, particularly in models involving differential equations (Sec. 11.3, Chap. 12). If \( f(x) \) is even \([f(-x) = f(x)]\) or odd \([f(-x) = -f(x)]\), they reduce to Fourier cosine or Fourier sine series, respectively (Sec. 11.2). If \( f(x) \) is given for \( 0 \leq x \leq L \) only, it has two half-range expansions of period \( 2L \), namely, a cosine and a sine series (Sec. 11.2).

The set of cosine and sine functions in (1) is called the trigonometric system. Its most basic property is its orthogonality on an interval of length \( 2L \); that is, for all integers \( m \) and \( n \neq m \) we have

\[
\int_{-L}^{L} \cos \frac{m\pi x}{L} \cos \frac{n\pi x}{L} \, dx = 0, \quad \int_{-L}^{L} \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L} \, dx = 0
\]

and for all integers \( m \) and \( n \),

\[
\int_{-L}^{L} \cos \frac{m\pi x}{L} \sin \frac{n\pi x}{L} \, dx = 0.
\]

This orthogonality was crucial in deriving the Euler formulas (2).
Partial sums of Fourier series minimize the square error (Sec. 11.4).
Replacing the trigonometric system in (1) by other orthogonal systems first leads to Sturm–Liouville problems (Sec. 11.5), which are boundary value problems for ODEs. These problems are eigenvalue problems and as such involve a parameter $\lambda$ that is often related to frequencies and energies. The solutions to Sturm–Liouville problems are called eigenfunctions. Similar considerations lead to other orthogonal series such as Fourier–Legendre series and Fourier–Bessel series classified as generalized Fourier series (Sec. 11.6).
Ideas and techniques of Fourier series extend to nonperiodic functions $f(x)$ defined on the entire real line; this leads to the Fourier integral

$$f(x) = \int_{0}^{\infty} [A(w) \cos wx + B(w) \sin wx] dw \quad \text{(Sec. 11.7)}$$

where

$$A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos wv dv, \quad B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin wv dv$$

or, in complex form (Sec. 11.9),

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w)e^{iwx} dw \quad (i = \sqrt{-1})$$

where

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-iwx} dx.$$

Formula (6) transforms $f(x)$ into its Fourier transform $\hat{f}(w)$, and (5) is the inverse transform.
Related to this are the Fourier cosine transform (Sec. 11.8)

$$\hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cos wx \, dx$$

and the Fourier sine transform (Sec. 11.8)

$$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \sin wx \, dx.$$

The discrete Fourier transform (DFT) and a practical method of computing it, called the fast Fourier transform (FFT), are discussed in Sec. 11.9.
The simple proof of this important theorem is quite similar to that of Theorem 1 in Sec. 2.1 and is left to the student.

Verification of solutions in Probs. 2–13 proceeds as for ODEs. Problems 16–23 concern PDEs solvable like ODEs. To help the student with them, we consider two typical examples.

**EXAMPLE 2**

**Solving** \( u_{xx} - u = 0 \) **Like an ODE**

Find solutions \( u \) of the PDE \( u_{xx} - u = 0 \) depending on \( x \) and \( y \).

**Solution.** Since no \( y \)-derivatives occur, we can solve this PDE like \( u'' - u = 0 \). In Sec. 2.2 we would have obtained \( u = Ae^x + Be^{-x} \) with constant \( A \) and \( B \). Here \( A \) and \( B \) may be functions of \( y \), so that the answer is

\[
u(x, y) = (A(x)e^y + B(x)e^{-y})
\]

with arbitrary functions \( A \) and \( B \). We thus have a great variety of solutions. Check the result by differentiation.

**EXAMPLE 3**

**Solving** \( u_{xy} = -u \) **Like an ODE**

Find solutions \( u = u(x, y) \) of this PDE.

**Solution.** Setting \( u_x = p \), we have \( p_y = -p \), \( p_y/y = -1 \), \( \ln |p| = -y + c(x) \), \( p = e^{c(x)e^{-y}} \) and by integration with respect to \( x \),

\[
u(x, y) = f(x)e^{-y} + g(y)
\]

where \( f(x) = \int c(x) \, dx \).

Here, \( f(x) \) and \( g(y) \) are arbitrary.

**PROBLEM SET 12.1**

1. **Fundamental theorem.** Prove it for second-order PDEs in two and three independent variables. Hint. Prove it by substitution.

2–13 **VERIFICATION OF SOLUTIONS**

Verify (by substitution) that the given function is a solution of the PDE. Sketch or graph the solution as a surface in space.

2–5 **Wave Equation (1) with suitable \( c \)**

2. \( u = x^2 + y^2 \)
3. \( u = \cos 4\pi \sin 2x \)
4. \( u = \sin kct \cos kx \)
5. \( u = \sin \alpha t \sin \beta x \)

6–9 **Heat Equation (2) with suitable \( c \)**

6. \( u = e^{-t} \sin x \)
7. \( u = e^{-r} \cos \omega t \cos \omega x \)
8. \( u = e^{-r} \sin \omega t \sin \omega x \)
9. \( u = e^{-\pi^2 \alpha^2 t} \cos 25x \)

10–13 **Laplace Equation (3)**

10. \( u = e^x \cos y, e^x \sin y \)
11. \( u = \arctan (y/x) \)
12. \( u = \cos y \sin x, \sin y \cosh x \)

13. \( u = x/(x^2 + y^2), y/(x^2 + y^2) \)

14. **TEAM PROJECT. Verification of Solutions**

(a) **Wave equation.** Verify that \( u(x, t) = v(x + ct) + w(x - ct) \) with any twice differentiable functions \( v \) and \( w \) satisfies \( (1) \).

(b) **Poisson equation.** Verify that each \( u \) satisfies \( (4) \) with \( f(x, y) \) as indicated.

\[
u = y/x \quad f = 2y/x^2
\]

\[
u = \sin xy \quad f = (x^2 + y^2) \sin xy
\]

\[
u = e^{x^2 - y^2} \quad f = 4(x^2 + y^2)e^{x^2 - y^2}
\]

\[
u = 1/\sqrt{x^2 + y^2} \quad f = (x^2 + y^2)^{-3/2}
\]

(c) **Laplace equation.** Verify that

\[
u = 1/\sqrt{x^2 + y^2} \quad \text{satisfies (6) and}
\]

\[
u = \ln (x^2 + y^2) \quad \text{satisfies (3). Is } u = 1/\sqrt{x^2 + y^2} \text{ a solution of (3)? Of what Poisson equation?}
\]

(d) Verify that \( u \) with any (sufficiently often differentiable) \( v \) and \( w \) satisfies the given PDE.

\[
u = v(x) + w(y) \quad u_{xx} = 0
\]

\[
u = v(x)w(y) \quad u_{xx} = u_{yy}
\]

\[
u = v(x + 2t) + w(x - 2t) \quad u_{tt} = 4u_{xx}
\]

15. **Boundary value problem.** Verify that the function \( u(x, y) = u \ln (x^2 + y^2) + b \) satisfies Laplace’s equation
(3) and determine $a$ and $b$ so that $u$ satisfies the
boundary conditions $u = 110$ on the circle
$x^2 + y^2 = 1$ and $u = 0$ on the circle $x^2 + y^2 = 100$.

16–23 PDEs SOLVABLE AS ODEs
This happens if a PDE involves derivatives with respect to
one variable only (or can be transformed to such a form),
so that the other variable(s) can be treated as parameter(s).
Solve for $u = u(x, y)$:
\begin{align*}
16. u_{yy} &= 0 & 17. u_{xx} + 16\pi^2 u &= 0 \\
18. 25u_{yy} - 4u &= 0 & 19. u_y + y^2 u &= 0 \\
20. 2u_{xx} + 9u_x + 4u &= -3 \cos x - 29 \sin x \\
21. u_{yy} + 6u_y + 13u &= 4e^y \\
22. u_{xy} &= u_x & 23. x^2u_{xx} + 2u_x - 2u &= 0 \\
24. Surface of revolution. Show that the solutions $z = z(x, y)$ of $yz_x = xz_y$ represent surfaces of revolution. Give examples. Hint. Use polar coordinates $r, \theta$ and show that the equation becomes $z_{rr} = 0$. \\
25. System of PDEs. Solve $u_{xx} = 0, u_{yy} = 0$
\end{align*}

12.2 Modeling: Vibrating String, Wave Equation

In this section we model a vibrating string, which will lead to our first important PDE,
that is, equation (3) which will then be solved in Sec. 12.3. The student should pay very
close attention to this delicate modeling process and detailed derivation starting from
scratch, as the skills learned can be applied to modeling other phenomena in general and
in particular to modeling a vibrating membrane (Sec. 12.7).

We want to derive the PDE modeling small transverse vibrations of an elastic string, such
as a violin string. We place the string along the $x$-axis, stretch it to length $L$, and fasten it
at the ends $x = 0$ and $x = L$. We then distort the string, and at some instant, call it $t = 0$,
we release it and allow it to vibrate. The problem is to determine the vibrations of the string,
that is, to find its deflection at any point $x$ and at any time $t > 0$; see Fig. 286.
$u(x, t)$ will be the solution of a PDE that is the model of our physical system to be
derived. This PDE should not be too complicated, so that we can solve it. Reasonable
simplifying assumptions (just as for ODEs modeling vibrations in Chap. 2) are as follows.

Physical Assumptions
\begin{enumerate}
  \item The mass of the string per unit length is constant (“homogeneous string”). The string
        is perfectly elastic and does not offer any resistance to bending.
  \item The tension caused by stretching the string before fastening it at the ends is so large
        that the action of the gravitational force on the string (trying to pull the string down a
        little) can be neglected.
  \item The string performs small transverse motions in a vertical plane; that is, every
        particle of the string moves strictly vertically and so that the deflection and the slope
        at every point of the string always remain small in absolute value.
\end{enumerate}

Under these assumptions we may expect solutions $u(x, t)$ that describe the physical
reality sufficiently well.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{deflected_string}
\caption{Deflected string at fixed time $t$. Explanation on p. 544}
\end{figure}