10. Appendix E - Continuous Equations

10.1 Auxiliary Equations

Effective Density of States:

Bulk: $$N_{c,v}(x) = \frac{1}{4p^{\frac{3}{2}}}\left(\frac{2m^{*}_{n,p}(x)kT_{n,p}(x)}{\hbar^{2}}\right)^{\frac{3}{2}}$$

QW: $$N_{c,vqw} = \frac{m^{*}_{n,p}kT_{n,p}}{p\hbar^{2}}$$

Quantum Well Energy Level:

$$\sqrt{\frac{2m^{*}_{qwn,p}E_{qwn,p}}{\hbar^{2}}}\tan\left(\sqrt{\frac{2m^{*}_{qwn,p}E_{qwn,p}}{\hbar^{2}}}\frac{L_{qw}}{2}\right)-\sqrt{\frac{2m^{*}_{bulkn,p}\left(V_{qwn,p}-E_{qwn,p}\right)}{\hbar^{2}}}=0$$

Quantum Well Wavefunction:

$$y_{n,p}(x) = \sqrt{\frac{2}{L_{qw}}}\sin\left(\frac{px}{L_{qw}}\right)$$

Tunneling Probability:

$$T_n(E) = e^{-\frac{2p}{\hbar}\int_0^{L(E)}\sqrt{2m^{*}_{n}(E_c-E)}dx} \qquad T_p(E) = e^{-\frac{2p}{\hbar}\int_0^{L(E)}\sqrt{2m^{*}_{p}(E-E_v)}dx}$$

Carrier Concentrations:

Free Carriers (Bulk): $n(x) = N_c F_{1/2}(h_c)$ $\qquad p(x) = N_v F_{1/2}(h_v)$

Bound Carriers (QW):

$$n_{2d} = N_{cqw}F_0\left(h_c - \tfrac{E_{qwn}}{kT_n}\right) \qquad p_{2d} = N_{vqw}F_0\left(h_v - \tfrac{E_{qwp}}{kT_p}\right)$$

$$n_b(x) = \left|y_n(x)\right|^2 n_{2d} \qquad p_b(x) = \left|y_p(x)\right|^2 p_{2d}$$

Free Carriers (QW-Boltzmann):

$$n_f(x) = e^{\eta_c}\left[N_c(x)\left(1 - \frac{\gamma\left(\frac{3}{2}, \frac{E_{c,top}}{kT_n}\right)}{\sqrt{\pi}/2}\right) - \frac{N_{cqw}}{L_{qw}} e^{-\frac{E_{c,top}}{kT_n}}\right]$$

$$p_f(x) = e^{\eta_v}\left[N_v(x)\left(1 - \frac{\gamma\left(\frac{3}{2}, \frac{E_{v,top}}{kT_p}\right)}{\sqrt{\pi}/2}\right) - \frac{N_{vqw}}{L_{qw}} e^{-\frac{E_{v,top}}{kT_p}}\right]$$

Carrier Energy Density:

Free Carriers (Bulk): $u_n = n\left(\frac{3}{2} kT_n \frac{F_{3/2}(\eta_c)}{F_{1/2}(\eta_c)} + E_c\right)$

$$u_p = p\left(\frac{3}{2} kT_p \frac{F_{3/2}(\eta_v)}{F_{1/2}(\eta_v)} - E_v\right)$$

Bound Carriers (QW):

$$u_{n,2d} = n_{2d}\left(kT_n \frac{F_1\left(\eta_c - \frac{E_{nqw}}{kT_n}\right)}{F_0\left(\eta_c - \frac{E_{nqw}}{kT_n}\right)} + E_{nqw} + E_c\right)$$

$$u_{p,2d} = p_{2d}\left(kT_p \frac{F_1\left(\eta_v - \frac{E_{pqw}}{kT_p}\right)}{F_0\left(\eta_v - \frac{E_{pqw}}{kT_p}\right)} + E_{pqw} - E_v\right)$$

$$u_n(x) = |\psi_n(x)|^2 u_{n,2d} \qquad u_p(x) = |\psi_p(x)|^2 u_{p,2d}$$

Total Charge:

$$\rho(x) = q(p(x) - n(x) + N_D^+(x) - N_A^-(x))$$

Ionized Doping Concentration:

$$N_D^+(\eta_c) = \frac{N_D}{1 + g_D e^{E_D/kT_n + \eta_c}} \qquad N_A^+(\eta_v) = \frac{N_A}{1 + g_A e^{E_A/kT_p + \eta_v}}$$

Charge Neutral Planck Potentials:

$$p(-\eta_{c0} - E_g/kT_L) - n(\eta_{c0}) + N_D^+(\eta_{c0}) - N_A^-(-\eta_{c0} - E_g/kT_L) = 0$$

$$\eta_{v0}(x) = -\eta_{c0}(x) - E_g(x)/kT_L$$

Charge Neutral Potential:

$$f(x) = -kT_L h_{c0}(0) + c(0) - c(x) + kT_L h_{c0}(x)$$

Built-in Potential:

$$f_0 = -kT_L h_{c0}(0) + c(0) - c(L) + kT_L h_{c0}(L)$$

Equilibrium Planck Potentials:

$$h_c(x) = \frac{f(x) + c(x) - c(0)}{kT_L} + h_c(0)$$

$$h_v(x) = -h_c(x) - E_g(x) / kT_L$$

Intrinsic Carrier Concentration:

Bulk: $$n_i^2 = N_c(T_L) N_v(T_L) e^{-E_g/kT_L} \frac{F_{1/2}(h_{c0})}{e^{h_{c0}}} \frac{F_{1/2}(h_{v0})}{e^{h_{v0}}}$$

QW: $$n_{i,2d}^2 = N_{cqw}(T_L) N_{vqw}(T_L) e^{-E_{gqw}/kT_L} \frac{F_0\left(h_{c0} - \frac{E_{qwn}}{kT_n}\right)}{e^{h_{c0} - \frac{E_{qwn}}{kT_n}}} \frac{F_0\left(h_{v0} - \frac{E_{qwp}}{kT_p}\right)}{e^{h_{v0} - \frac{E_{qwp}}{kT_p}}}$$

Complex Refractive Index and Impedance

$$n_j = n_{real} - i\frac{a}{2k_0} \qquad Z_j = \frac{Z_0}{n_j}$$

Electromagnetic Fields:

$$\begin{bmatrix} E_{j+1}^+ \\ E_{j+1}^- \end{bmatrix} = \frac{1}{2Z_j} \begin{bmatrix} \left(Z_j + Z_{j+1}\right) e^{-ik_0 n_{j+1}\left(x_{j+1} - x_j\right)} & \left(Z_j - Z_{j+1}\right) e^{-ik_0 n_{j+1}\left(x_{j+1} - x_j\right)} \\ \left(Z_j - Z_{j+1}\right) e^{ik_0 n_{j+1}\left(x_{j+1} - x_j\right)} & \left(Z_j + Z_{j+1}\right) e^{ik_0 n_{j+1}\left(x_{j+1} - x_j\right)} \end{bmatrix} \begin{bmatrix} E_j^+ \\ E_j^- \end{bmatrix}$$

$$\begin{bmatrix} E_j^+ \\ E_j^- \end{bmatrix} = \frac{1}{2Z_{j+1}} \begin{bmatrix} \left(Z_{j+1} + Z_j\right) e^{ik_0 n_{j+1}\left(x_{j+1} - x_j\right)} & \left(Z_{j+1} - Z_j\right) e^{-ik_0 n_{j+1}\left(x_{j+1} - x_j\right)} \\ \left(Z_{j+1} - Z_j\right) e^{ik_0 n_{j+1}\left(x_{j+1} - x_j\right)} & \left(Z_{j+1} + Z_j\right) e^{-ik_0 n_{j+1}\left(x_{j+1} - x_j\right)} \end{bmatrix} \begin{bmatrix} E_{j+1}^+ \\ E_{j+1}^- \end{bmatrix}$$

Poynting Vectors:

$$\left\langle S_j \right\rangle = \left\langle S_j^+ \right\rangle - \left\langle S_j^- \right\rangle + \Re\left[\frac{i\Im\left(E_j^- E_j^{+*} e^{i\left(2k_0 \Re\left(n_j\right)\left(x - x_j\right)\right)} \right)}{Z_j^*} \right]$$

$$\left\langle S_j^+ \right\rangle = \tfrac{1}{2}\left| E_j^+ \right|^2 e^{-a\left(x - x_j\right)} \Re\left(\tfrac{1}{Z_j^*} \right) \qquad \left\langle S_j^- \right\rangle = \tfrac{1}{2}\left| E_j^- \right|^2 e^{a\left(x - x_j\right)} \Re\left(\tfrac{1}{Z_j^*} \right)$$

Stimulated emission energy levels:

Bulk:

$$E_{n,stim} - E_c = \frac{m_p^*}{m_n^* + m_p^*}\left(hu_{stim} - E_g \right)$$

$$E_v - E_{p,stim} = \frac{m_n^*}{m_n^* + m_p^*}\left(hu_{stim} - E_g \right)$$

QW:

$$E_{n,stim} - E_c = \frac{m_p^*}{m_n^* + m_p^*}\left(hu_{stim} - E_{gqw} \right) + E_{qwn}$$

$$E_v - E_{p,stim} = \frac{m_n^*}{m_n^* + m_p^*}\left(hu_{stim} - E_{gqw} \right) + E_{qwp}$$

Local gain:

$$g(hu_{stim}) = a\,(hu_{stim})\left[f\left(E_{n,stim}\right) - f\left(E_{p,stim}\right) \right]$$

Richardson Constant:

$$A_{n,p}^* = \tfrac{q m_{n,p}^* k^2}{2 p^2 \hbar^3} \qquad \text{(use lighter mass)}$$

Carrier Recombination/Generation Rates:

Spontaneous band-to band:

Bulk: $U_{b-b} = B(np - n_i^2)$

QW: $U_{b-b} = \dfrac{1}{L_{qw}} \left| \displaystyle\int_0^{L_{qw}} y_n(x) y_p(x)\, dx \right|^2 B_{qw}(n_{2d} p_{2d} - n_{i,2d}^2)$

Shockley-Hall-Reed:

$$\text{Bulk:} \quad U_{shr} = \frac{np - n_i^2}{(p+n_i)t_n + (n+n_i)t_p}$$

$$\text{QW:} \quad U_{shr} = \frac{1}{L_{qw}} \left| \int_0^{L_{qw}} y_n(x) y_p(x) dx \right|^2 \frac{n_{2d} p_{2d} - n_{i,2d}^2}{(p_{2d} + n_{i,2d})t_{no} + (n_{2d} + n_{i,2d})t_{po}}$$

External Optical Generation:

$$G_{opt}(x) = \sum_{hu_{opt}} \left(\frac{a\left\langle S_j^+ \right\rangle + a\left\langle S_j^- \right\rangle - k_0 \Re\left(n_j\right) \Re\left[\frac{2i\Re\left(E_j^- E_j^{+*}\right)}{Z^*}\right]}{hu_{opt}} \right)$$

Stimulated Emission:

$$U_{stim}(x) = v_{stim} g(x) S \left|\mathcal{E}_{stim}(x)\right|^2 \qquad A_c \int_0^L \left|\mathcal{E}_{stim}(x)\right|^2 dx = 1$$

Total photon emission rates:

$$\tilde{U}_{b-b} = A_c \int_0^L U_{b-b}(x) dx \qquad \tilde{U}_{stim} = A_c \int_0^L U_{stim}(x) dx$$

Distributed mirror loss and photon lifetime:

$$a_m = \frac{1}{2L_c} \ln\left(\frac{1}{R_l R_r}\right) \qquad t_{ph} = \frac{1}{v_{stim}\left(a_m + a_s\right)}$$

Carrier Energy Loss Rates:

$$W_{n,shr}^{tot} = \left(\frac{u_n}{n}\right) U_{shr} \qquad W_{p,shr}^{tot} = \left(\frac{u_p}{p}\right) U_{shr}$$

$$W_{n,b-b}^{tot} = \left(\frac{u_n}{n}\right) U_{b-b} \qquad W_{p,b-b}^{tot} = \left(\frac{u_p}{p}\right) U_{b-b}$$

$$W_{n,stim}^{tot} = E_{n,stim} U_{stim} \qquad W_{p,stim}^{tot} = E_{p,stim} U_{stim}$$

Bulk:

$$W_{n,opt}^{tot} = \sum_{hu_{opt}} \left(\tfrac{m_p^*}{m_n^*+m_p^*} (hu_{opt} - E_g) G_{u,opt} \right) + E_c G_{opt}$$
$$W_{p,opt}^{tot} = \sum_{hu_{opt}} \left(\tfrac{m_n^*}{m_n^*+m_p^*} (hu_{opt} - E_g) G_{u,opt} \right) - E_v G_{opt}$$

QW:

$$W_{n,opt}^{tot} = \sum_{hu_{opt}} \left(\tfrac{m_p^*}{m_n^*+m_p^*} (hu_{opt} - E_{gqw}) G_{u,opt} \right) + (E_{nqw} + E_c) G_{opt}$$
$$W_{p,opt}^{tot} = \sum_{hu_{opt}} \left(\tfrac{m_n^*}{m_n^*+m_p^*} (hu_{opt} - E_{gqw}) G_{u,opt} \right) + (E_{pqw} - E_v) G_{opt}$$

$$W_{n,relax} = \frac{u_n(T_n) - u_n(T_{lat})}{t_{wn}} \qquad W_{p,relax} = \frac{u_p(T_p) - u_p(T_{lat})}{t_{wp}}$$

Total Carrier Energy Loss:

$$W_n^{tot} = W_{n,shr}^{tot} + W_{n,b-b}^{tot} + W_{n,stim}^{tot} - W_{n,opt}^{tot} + W_{n,relax}$$

$$W_p^{tot} = W_{p,shr}^{tot} + W_{p,b-b}^{tot} + W_{p,stim}^{tot} - W_{p,opt}^{tot} + W_{p,relax}$$

$$W_{lat}^{tot} = W_{n,shr}^{tot} + W_{p,shr}^{tot} + W_{n,relax} + W_{p,relax}$$

Effective Environment Thermal Conductivity:

$$k_e(x) = k(x) \frac{2p}{\ln\left(\frac{r_e}{r_d}\right)}$$

10.2 Rate Equations

Poisson's Equation: $\nabla \bullet \mathbf{D}(x) - r(x) = 0$

Electron Rate Equation: $\nabla \bullet (\mathbf{J_n}(x)/q) - U_{tot}(x) = 0$

Hole Rate Equation: $\nabla \bullet (\mathbf{J_p}(x)/q) + U_{tot}(x) = 0$

Photon Rate Equation: $\widetilde{U}_{stim} + b\widetilde{U}_{b-b} - S/t_{ph} = 0$

Electron Energy Rate Equation $\nabla \bullet \mathbf{S}_n^{tot}(x) + W_n^{tot}(x) = 0$

Hole Energy Rate Equation $\nabla \bullet \mathbf{S}_p^{tot}(x) + W_p^{tot}(x) = 0$

Lattice Energy Rate Equation $\nabla \bullet \mathbf{S}_{lat}(x) - W_{lat}(x) = 0$

Total Energy Rate Equation (for T_n=T_p=T_{lat}=T):

$$\nabla \bullet (\mathbf{S}_{lat}(x) + \mathbf{S}_n(x) + \mathbf{S}_p(x)) + W_n^{tot} + W_p^{tot} - W_{lat}^{tot} = 0$$

10.3 Vector Equations

Displacement:

$$\mathbf{D}(x) = - \in (x) \nabla f(x)$$

Drift-Diffusion Current:

$$\mathbf{J_n} = m_n n \left\{ \left(kT_n \nabla h_c + \nabla E_c \right) + \left(\tfrac{5}{2} + u_n \right) \frac{F_{3/2+u_n}(h_c)}{F_{1/2+u_n}(h_c)} \nabla kT_n \right\}$$

$$\mathbf{J_n} = m_n kT_n \frac{F_{1/2}(h_c)}{F_{-1/2}(h_c)} \nabla n + m_n n \nabla E_c - \frac{3kT_n}{2} m_n \frac{F_{1/2}(h_c)}{F_{-1/2}(h_c)} n \nabla \ln(m_n^*)$$
$$+ m_n n \left(\left(\tfrac{5}{2} + u_n \right) \frac{F_{3/2+u_n}(h_c)}{F_{1/2+u_n}(h_c)} - \frac{3}{2} \frac{F_{1/2}(h_c)}{F_{-1/2}(h_c)} \right) \nabla kT_n$$

$$\mathbf{J}_p = -m_p p \left\{ \left(kT_p \nabla h_v - \nabla E_v \right) + \left(\tfrac{5}{2} + u_p \right) \frac{F_{3/2+u_p}(h_v)}{F_{1/2+u_p}(h_v)} \nabla kT_p \right\}$$

$$\mathbf{J}_p = -m_p kT_p \frac{F_{1/2}(h_v)}{F_{-1/2}(h_v)} \nabla p + m_p p \nabla E_v + \frac{3kT_p}{2} m_p \frac{F_{1/2}(h_v)}{F_{-1/2}(h_v)} p \nabla \ln(m_p^*)$$
$$- m_p p \left(\left(\tfrac{5}{2} + u_p \right) \frac{F_{3/2+u_p}(h_v)}{F_{1/2+u_p}(h_v)} - \frac{3}{2} \frac{F_{1/2}(h_v)}{F_{-1/2}(h_v)} \right) \nabla kT_p$$

Thermionic Emission Current:

$$\mathbf{J}_n^{therm} = \mathbf{J}_{n-\to+}^{therm} - \mathbf{J}_{n+\to-}^{therm}$$

Fermi-Dirac: $$\mathbf{J}_{n-\to+}^{therm} = -A_n^* T_{n-}^2 \left\{ \mathrm{Li}_2\left(\tfrac{1}{1+e^{U_{c-}/kT_{n-}-h_{c-}}}\right) + \tfrac{1}{2}\left[\ln\left(1+e^{h_{c-}-U_{c-}/kT_{n-}}\right)\right]^2 \right\}$$

Boltzmann: $$\mathbf{J}_{n-\to+}^{therm} = -A_n^* T_{n-}^2 e^{h_{c-}-U_{c-}/kT_{n-}}$$

$$\mathbf{J}_p^{therm} = \mathbf{J}_{p-\to+}^{therm} - \mathbf{J}_{p+\to-}^{therm}$$

Fermi-Dirac: $$\mathbf{J}_{p-\to+}^{therm} = A_p^* T_{p-}^2 \left\{ \mathrm{Li}_2\left(\tfrac{1}{1+e^{U_{v-}/kT_{p-}-h_{v-}}}\right) + \tfrac{1}{2}\left[\ln\left(1+e^{h_{v-}-U_{v-}/kT_{p-}}\right)\right]^2 \right\}$$

Boltzmann: $$\mathbf{J}_{p-\to+}^{therm} = A_p^* T_{p-}^2 e^{h_{v-}-U_{v-}/kT_{p-}}$$

Tunneling Current:

$$\mathbf{J}_n^{tun} = \mathbf{J}_{n-\to+}^{tun} - \mathbf{J}_{n+\to-}^{tun}$$

Fermi-Dirac: $$\mathbf{J}_{n-\to+}^{tun} = -A_n^* \frac{T_{n-}}{k} \int_0^{U_{c-}} T_n(E_x) \ln\left(1+e^{h_c - E_x/kT_n}\right) dE_x$$

Boltzmann: $$\mathbf{J}_{n-\to+}^{tun} = -A_n^* \frac{T_{n-}}{k} \int_0^{U_{c-}} T_n(E_x) e^{h_c - E_x/kT_n} dE_x$$

$$\mathbf{J}_p^{tun} = \mathbf{J}_{p-\to+}^{tun} - \mathbf{J}_{p+\to-}^{tun}$$

Fermi-Dirac: $$\mathbf{J}_{p-\to+}^{tun} = A_p^* \frac{T_{p-}}{k} \int_0^{U_{v-}} T_p(E_x) \ln\left(1+e^{h_v - E_x/kT_p}\right) dE_x$$

Boltzmann: $$\mathbf{J}_{p-\to+}^{tun} = A_p^* \frac{T_{p-}}{k} \int_0^{U_{v-}} T_p(E_x) e^{h_v - E_x/kT_p} dE_x$$

Lattice Energy Flux:

$$\mathbf{S}_{lat} = -k(x)\nabla T_L \hat{\mathbf{x}} + \frac{\Delta x}{p r_d^2} k_e(x)(T_L(x) - T_{env})\hat{\mathbf{r}}$$

Drift Diffusion Energy Flux:

$$\mathbf{S}_n^{tot} = -m_n n \frac{kT_n}{q} \left\{ \begin{array}{l} \left(kT_n \nabla h_c + \nabla E_c\right)\left(\frac{5}{2} + u_n\right) \frac{F_{3/2+u_n}(h_c)}{F_{1/2+u_n}(h_c)} \\ +\nabla kT_n \left(\frac{7}{2} + u_n\right)\left(\frac{5}{2} + u_n\right) \frac{F_{5/2+u_n}(h_c)}{F_{1/2+u_n}(h_c)} \end{array} \right\} + \frac{E_c \mathbf{J}_n}{-q}$$

$$\mathbf{S}_n^{tot} = -\left(\frac{5}{2} + u_n\right) \frac{F_{3/2+u_n}(h_c)}{F_{1/2+u_n}(h_c)} m_n \frac{kT_n}{q} \left\{ \begin{array}{l} kT_n \frac{F_{1/2}(h_c)}{F_{-1/2}(h_c)} \nabla n + n \nabla E_c \\ -\frac{3}{2} kT_n n \frac{F_{1/2}(h_c)}{F_{-1/2}(h_c)} \nabla \ln(m_n^*) \\ +n\left(\left(\frac{7}{2} + u_n\right) \frac{F_{5/2+u_n}(h_c)}{F_{3/2+u_n}(h_c)} - \frac{3}{2} \frac{F_{1/2}(h_c)}{F_{-1/2}(h_c)}\right) \nabla kT_n \end{array} \right\} + \frac{E_c \mathbf{J}_n}{-q}$$

$$\mathbf{S}_p^{tot} = -m_p p \frac{kT_p}{q} \left\{ \begin{array}{l} \left(kT_p \nabla h_v - \nabla E_v\right)\left(\frac{5}{2} + u_p\right) \frac{F_{3/2+u_p}(h_v)}{F_{1/2+u_p}(h_v)} \\ +\nabla kT_p \left(\frac{7}{2} + u_p\right)\left(\frac{5}{2} + u_p\right) \frac{F_{5/2+u_p}(h_v)}{F_{1/2+u_p}(h_v)} \end{array} \right\} - \frac{E_v \mathbf{J}_p}{q}$$

$$\mathbf{S}_p^{tot} = -\left(\frac{5}{2} + u_p\right) \frac{F_{3/2+u}(h_v)}{F_{1/2+u}(h_v)} m_p \frac{kT_p}{q} \left\{ \begin{array}{l} kT_p \frac{F_{1/2}(h_v)}{F_{-1/2}(h_v)} \nabla p - p \nabla E_v \\ -\frac{3}{2} kT_p p \frac{F_{1/2}(h_v)}{F_{-1/2}(h_v)} \nabla \ln(m_p^*) \\ +p\left(\left(\frac{7}{2} + u_p\right) \frac{F_{5/2+u_p}(h_v)}{F_{3/2+u_p}(h_v)} - \frac{3}{2} \frac{F_{1/2}(h_v)}{F_{-1/2}(h_v)}\right) \nabla kT_p \end{array} \right\} - \frac{E_v \mathbf{J}_p}{q}$$

Thermionic Emission Energy Flux:

$$\mathbf{S}_n^{therm} = \mathbf{S}_{n-\to+}^{therm} + \frac{E_{c-} \mathbf{J}_{n-\to+}^{therm}}{-q} - \mathbf{S}_{n+\to-}^{therm} - \frac{E_{c+} \mathbf{J}_{n+\to-}^{therm}}{-q}$$

Fermi-Dirac:

$$\mathbf{S}_{n-\to+}^{therm} = \frac{A_n^* kT_{n-}^3}{q} \left[\begin{array}{l} 2\left\{ \mathrm{Li}_3\left(\frac{1}{1+e^{U_{c-}/kT_{n-} - h_{c-}}} \right) + \mathrm{Li}_3\left(\frac{1}{1+e^{h_{c-} - U_{c-}/kT_{n-}}} \right) - \mathrm{Li}_3(1) \right\} \\ +\left(\frac{U_{c-}}{kT_{n-}}\right) \mathrm{Li}_2\left(\frac{1}{1+e^{U_{c-}/kT_{n-} - h_{c-}}} \right) \\ +\frac{1}{3} h_{c-}\left(h_{c-}^2 + p^2\right) - \left(\frac{U_{c-}}{kT_{n-}}\right)\left(\frac{h_{c-}^2}{2} + \frac{p^2}{3}\right) + \frac{1}{6}\left(\frac{U_{c-}}{kT_{n-}}\right)^3 \\ -\ln\left(1+e^{U_{c-}/kT_{n-} - h_{c-}}\right) \left\{ \begin{array}{l} \frac{2}{3} \ln^2\left(1+e^{U_{c-}/kT_{n-} - h_{c-}}\right) \\ -\left(\frac{3}{2} \frac{U_{c-}}{kT_{n-}} - h_{c-}\right) \ln\left(1+e^{U_{c-}/kT_{n-} - h_{c-}}\right) \\ +\left(\frac{U_{c-}}{kT_{n-}}\left(\frac{U_{c-}}{kT_{n-}} - h_{c-}\right) - \frac{p^2}{3}\right) \end{array} \right\} \end{array} \right]$$

Boltzmann: $$\mathbf{S}_{n-\to+}^{therm} = \frac{A_n^* k T_{n-}^3}{q}\left(2+\frac{U_{c-}}{kT_{n-}}\right)e^{h_{c-}-U_{c-}/kT_{n-}}$$

$$\mathbf{S}_p^{therm} = \mathbf{S}_{p-\to+}^{therm} - \frac{E_{v-}\mathbf{J}_{p-\to+}^{therm}}{q} - \mathbf{S}_{p+\to-}^{therm} + \frac{E_{v+}\mathbf{J}_{p+\to-}^{therm}}{q}$$

Fermi-Dirac:

$$\mathbf{S}_{p-\to+}^{therm} = \frac{A_p^* k T_{p-}^3}{q}\left[\begin{array}{l} 2\left\{\mathrm{Li}_3\left(\frac{1}{1+e^{U_{v-}/kT_{p-}-h_{v-}}}\right)+\mathrm{Li}_3\left(\frac{1}{1+e^{h_{v-}-U_{v-}/kT_{p-}}}\right)-\mathrm{Li}_3(1)\right\} \\ +\left(\frac{U_{v-}}{kT_{p-}}\right)\mathrm{Li}_2\left(\frac{1}{1+e^{U_{v-}/kT_{p-}-h_{v-}}}\right) \\ +\frac{1}{3}h_{v-}\left(h_{v-}^2+p^2\right)-\left(\frac{U_{v-}}{kT_{p-}}\right)\left(\frac{h_{v-}^2}{2}+\frac{p^2}{3}\right)+\frac{1}{6}\left(\frac{U_{v-}}{kT_{p-}}\right)^3 \\ -\ln\left(1+e^{U_{v-}/kT_{p-}-h_{v-}}\right)\left\{\begin{array}{l} \frac{2}{3}\ln^2\left(1+e^{U_{v-}/kT_{p-}-h_{v-}}\right) \\ -\left(\frac{3}{2}\frac{U_{v-}}{kT_{p-}}-h_{v-}\right)\ln\left(1+e^{U_{v-}/kT_{p-}-h_{v-}}\right) \\ +\left(\frac{U_{v-}}{kT_{p-}}\left(\frac{U_{v-}}{kT_{p-}}-h_{v-}\right)-\frac{p^2}{3}\right) \end{array}\right\} \end{array}\right]$$

Boltzmann: $$\mathbf{S}_{p-\to+}^{therm} = \frac{A_p^* k T_{p-}^3}{q}\left(2+\frac{U_{v-}}{kT_{p-}}\right)e^{h_{v-}-U_{v-}/kT_{p-}}$$

Tunneling Energy Flux:

$$\mathbf{S}_n^{tun} = \mathbf{S}_{n-\to+}^{tun} + \frac{E_{c-}\mathbf{J}_{n-\to+}^{tun}}{-q} - \mathbf{S}_{n+\to-}^{tun} - \frac{E_{c+}\mathbf{J}_{n+\to-}^{tun}}{-q}$$

Fermi-Dirac:

$$\mathbf{S}_{n-\to+}^{tun} = \frac{A_n^* T_{n-}^2}{q}\int_{U_{c-}}^{\infty} T_n(E_x)\left\{\begin{array}{r} \left(h_c-\ln\left(1+e^{h_c-E_x/kT_{n-}}\right)\right)\ln\left(1+e^{h_c-E_x/kT_{n-}}\right)+\mathrm{Li}_2\left(\frac{1}{1+e^{E_x/kT_{n-}-h_c}}\right) \\ +\ln\left(\frac{1}{1+e^{E_x/kT_{n-}-h_c}}\right)\ln\left(\frac{1}{1+e^{h_c-E_x/kT_{n-}}}\right) \end{array}\right\}dE_x$$

Boltzmann: $$\mathbf{S}_{n-\to+}^{tun} = \frac{A_n^* T_{n-}^2}{q}\int_0^{U_{c-}} T_n(E_x)\left(1+\frac{E_x}{kT_{n-}}\right)e^{h_c-E_x/kT_{n-}}\,dE_x$$

$$\mathbf{S}_p^{tun} = \mathbf{S}_{p-\to+}^{tun} - \frac{E_{v-}\mathbf{J}_{p-\to+}^{tun}}{q} - \mathbf{S}_{p+\to-}^{tun} + \frac{E_{v+}\mathbf{J}_{p+\to-}^{tun}}{q}$$

Fermi-Dirac:

$$\mathbf{S}_{p-\to+}^{tun} = \frac{A_p^* T_{p-}^2}{q}\int_{U_{v-}}^{\infty} T_p(E_x)\left\{\begin{aligned}&\left(h_v - \ln\left(1+e^{h_v - E_x/kT_{p-}}\right)\right)\ln\left(1+e^{h_v-E_x/kT_{p-}}\right) + \mathrm{Li}_2\left(\frac{1}{1+e^{E_x/kT_{p-}-h_v}}\right)\\&+\ln\left(\frac{1}{1+e^{E_x/kT_{p-}-h_v}}\right)\ln\left(\frac{1}{1+e^{h_v-E_x/kT_{p-}}}\right)\end{aligned}\right\}dE_x$$

Boltzmann: $$\mathbf{S}_{p-\to+}^{tun} = \frac{A_p^* T_{p-}^2}{q}\int_0^{U_{v-}} T_p(E_x)\left(1+\frac{E_x}{kT_{p-}}\right)e^{h_v - E_x/kT_{p-}}dE_x$$

10.4 Boundary Conditions

Electrical:

$$f(0) = V_l \qquad f(L) = f_0 + V_r$$

$$h_c(0) = h_{c0}(0) \qquad h_c(L) = h_{c0}(L)$$

$$h_v(0) = h_{v0}(0) \qquad h_v(L) = h_{v0}(L)$$

Thermal:

$$T_n(0) = T_L(0) \qquad T_n(L) = T_L(L)$$

$$T_p(0) = T_L(0) \qquad T_p(L) = T_L(L)$$

$$-k(0)\nabla T_L\big|_{x=0} + k_l T_L(0) = k_l T_l$$
$$k(L)\nabla T_L\big|_{x=L} + k_r T_L(L) = k_r T_r$$

10.5 Normalization Constants

Table 11 - Normalization constants for Si and GaAs

Value	Symbol	Si Value	GaAs Value
Charge (C)	q	1.602×10^{-19}	1.602×10^{-19}
Temperature (K)	T_0	300	300
Concentration (cm^{-3})	$n_0 = \sqrt{N_c(T_0)N_v(T_0)}$	2.268×10^{19}	1.813×10^{18}
Potential (V)	$V_T = \frac{kT_0}{q}$	0.02586	0.02586
Permitivity (F cm^{-1})	$\in_0$	8.854×10^{-14}	8.854×10^{-14}
Length (cm)	$x_0 = \sqrt{\frac{\in_0 V_T}{q n_0}}$	2.510×10^{-8}	8.876×10^{-8}
Mobility (cm^2 V^{-1} s^{-1})	m_0	1	1
Time (s)	$t_0 = \frac{x_0^2}{m_0 V_T}$	2.437×10^{-14}	3.047×10^{-13}
Recombination Rate (cm^{-3} s^{-1})	$U_0 = \frac{n_0}{t_0}$	9.306×10^{32}	5.953×10^{30}
Thermal Conductivity (W K^{-1} cm^{-1})	$k_0 = \frac{qV_T n_0 x_0^2}{T_0 t_0}$	8.0966×10^{-6}	6.477×10^{-7}
Field (V cm^{-1})	$\mathbf{e}_0 = \frac{V_T}{x_0}$	1.030×10^{6}	2.913×10^{5}
Current (A cm^{-2})	$\mathbf{J}_0 = q m_0 n_0 \mathbf{e}_0$	3.743×10^{6}	8.465×10^{4}
Energy (eV)	E_0	0.02586	0.02586
Intensity (eV cm^{-2} s^{-1})	$I_0 = \frac{E_0 x_0 n_0}{t_0}$	6.041×10^{23}	1.367×10^{22}