12. Appendix G - Special Derivations

These derivations are for electrons only. The same techniques will yield the corresponding equations for holes. All temperatures are electron temperatures and all masses are electron density of states masses.

12.1 Derivation 1 - Free Carrier Quantum Well Concentration

$$n_f = \int_{E_c+E_{c,top}}^{\infty} \left[\tfrac{1}{2p^2} \left(\tfrac{2m}{\hbar^2} \right)^{\frac{3}{2}} \left(E - E_c \right)^{\frac{1}{2}} - \tfrac{m}{p\hbar^2 L_{qw}} \right] \frac{1}{1+e^{\left(E-E_{fn}\right)/kT}} dE$$

Assuming Boltzmann statistics and let:

$$h = (E - E_c)/kT \qquad h_c = (E_{fn} - E_c)/kT$$

yields:

$$n_f = e^{h_c} \int_{E_{c,top}/kT}^{\infty} \left[\tfrac{1}{2p^2} \left(\tfrac{2mkT}{\hbar^2} \right)^{\frac{3}{2}} h^{\frac{1}{2}} - \tfrac{mkT}{p\hbar^2 L_{qw}} \right] e^{-h} dh$$

$$n_f = e^{h_c} \left[\int_0^{\infty} \tfrac{2}{\sqrt{p}} N_c h^{\frac{1}{2}} e^{-h} dh - \int_0^{E_{c,top}/kT} \tfrac{2}{\sqrt{p}} N_c h^{\frac{1}{2}} e^{-h} dh - \int_{E_{c,top}/kT_n}^{\infty} \tfrac{N_{cqw}}{L_{qw}} e^{-h} dh \right]$$

$$n_f = e^{h_c} \left[N_c \left(1 - \frac{g(\frac{3}{2}, \frac{E_{c,top}}{kT})}{\sqrt{p}/2} \right) - \frac{N_{cqw}}{L_{qw}} e^{-\frac{E_{c,top}}{kT}} \right]$$

12.2 Derivation 2 - Fermi-Dirac Thermionic Emission Current

$$\mathbf{J}_{n-\rightarrow+}^{therm} = -q \int_{U_{c-}}^{\infty} \mathbf{v}_x g(E) f(E) dE$$

let:

$$h_c = (E_{fn} - E_c)/kT \qquad E - E_c = \tfrac{1}{2} m\mathbf{v}^2 \qquad dE = m\mathbf{v}d\mathbf{v}$$

then:

$$\mathbf{J}_{n-\to+}^{therm} = -q \tfrac{1}{2p^2}\left(\tfrac{2m}{\hbar^2}\right)^{\frac{3}{2}} \int_{U_{c-}}^{\infty} \frac{\mathbf{v}_x (E-E_c)^{\frac{1}{2}}}{1+e^{(E-E_c)/kT-h_c}} dE$$

$$\mathbf{J}_{n-\to+}^{therm} = -q \tfrac{1}{2p^2}\left(\tfrac{2m}{\hbar^2}\right)^{\frac{3}{2}} \sqrt{\tfrac{m}{2}} m \int_{\sqrt{\frac{2U_{c-}}{m}}}^{\infty} \frac{\mathbf{v}_x \mathbf{v}^2}{1+e^{m\mathbf{v}^2/2kT-h_c}} d\mathbf{v}$$

Convert from spherical to cylindrical coordinates:

$$\mathbf{J}_{n-\to+}^{therm} = -q \tfrac{1}{4p^2}\left(\tfrac{2m}{\hbar^2}\right)^{\frac{3}{2}} \sqrt{\tfrac{m}{2}} m \int_{\sqrt{\frac{2U_{c-}}{m}}}^{\infty} \int_{0}^{\infty} \frac{\mathbf{v}_x \mathbf{v}_{yz} e^{-m\mathbf{v}_{yz}^2/2kT}}{e^{-m\mathbf{v}_{yz}^2/2kT} + e^{m\mathbf{v}_x^2/2kT-h_c}} d\mathbf{v}_{yz} d\mathbf{v}_x$$

let:

$$y = e^{-m\mathbf{v}_{yz}^2/2kT} \qquad dy = -\tfrac{m\mathbf{v}_{yz}}{kT} e^{-m\mathbf{v}_{yz}^2/2kT} d\mathbf{v}_{yz}$$

then:

$$\mathbf{J}_{n-\to+}^{therm} = -q \tfrac{1}{4p^2}\left(\tfrac{2m}{\hbar^2}\right)^{\frac{3}{2}} \sqrt{\tfrac{m}{2}} m \int_{\sqrt{\frac{2U_{c-}}{m}}}^{\infty} \int_{1}^{0} \frac{-\frac{kT}{m}\mathbf{v}_x}{y + e^{m\mathbf{v}_x^2/2kT-h_c}} dy d\mathbf{v}_x$$

$$\mathbf{J}_{n-\to+}^{therm} = -q \tfrac{1}{4p^2}\left(\tfrac{2m}{\hbar^2}\right)^{\frac{3}{2}} \sqrt{\tfrac{m}{2}} kT \int_{\sqrt{\frac{2U_{c-}}{m}}}^{\infty} \mathbf{v}_x \ln\left(1+e^{h_c - m\mathbf{v}_x^2/2kT}\right) d\mathbf{v}_x$$

let:

$$E_x = \tfrac{1}{2} m\mathbf{v}_x^2 \qquad \mathbf{v}_x = \sqrt{\frac{2E_x}{m}} \qquad d\mathbf{v}_x = \frac{1}{\sqrt{2mE_x}} dE_x$$

then:

$$\mathbf{J}_{n-\to+}^{therm} = -A_n^* \frac{T}{k} \int_{U_{c-}}^{\infty} \ln\left(1+e^{h_c - E_x/kT}\right) dE_x$$

This is the equation for tunneling current (include transmission probability).

let:

$$A = \frac{E_x}{kT} - h_c \qquad E_x = kT(A + h_c) \qquad dE_x = kTdA$$

then:

$$\mathbf{J}_{n-\to+}^{therm} = -A_n^* T^2 \int_{\frac{U_{c-}}{kT} - h_c}^{\infty} \ln\left(1 + e^{-A}\right) dA$$

let:

$$z = -e^{-A} \qquad dz = e^{-A} dA$$

$$\mathbf{J}_{n-\to+}^{therm} = -A_n^* T^2 \int_0^{-e^{h_c - U_{c-}/kT}} \frac{\ln(1-z)}{z} dz$$

$$\mathbf{J}_{n-\to+}^{therm} = -A_n^* T^2 \, \mathrm{Li}_2\left(-e^{h_c - U_{c-}/kT}\right)$$

Considerable algebra yields:

$$\mathbf{J}_{n-\to+}^{therm} = -A_n^* T^2 \left\{ \mathrm{Li}_2\left(\tfrac{1}{1+e^{U_{c-}/kT - h_c}}\right) + \tfrac{1}{2}\left[\ln\left(1 + e^{h_c - U_{c-}/kT}\right)\right]^2 \right\}$$

12.3 Derivation 3 - Boltzmann Thermionic Emission Energy Flux

$$\mathbf{S}_{n-\to+}^{therm} = \int_{U_{c-}}^{\infty} \mathbf{v}_x (E - E_c) g(E) f(E) dE$$

let:

$$h_c = (E_{fn} - E_c)/kT \qquad E - E_c = \tfrac{1}{2} m\mathbf{v}^2 \qquad dE = m\mathbf{v}d\mathbf{v}$$

then assuming Boltzmann statistics:

$$\mathbf{S}_{n-\to+}^{therm} = \tfrac{1}{2p^2}\left(\tfrac{2m}{\hbar^2}\right)^{\frac{3}{2}} e^{h_c} \int_{U_{c-}}^{\infty} \mathbf{v}_x (E - E_c)^{\frac{3}{2}} e^{-(E-E_c)/kT} dE$$

$$\mathbf{S}_{n-\to+}^{therm} = \tfrac{1}{2p^2}\left(\tfrac{2m}{\hbar^2}\right)^{\frac{3}{2}} \left(\tfrac{m}{2}\right)^{\frac{3}{2}} m e^{h_c} \int_{\sqrt{\frac{2U_{c-}}{m}}}^{\infty} \mathbf{v}_x e^{-m\mathbf{v}^2/2kT} \mathbf{v}^4 d\mathbf{v}$$

Convert to cylindrical coordinates:

$$\mathbf{S}_{n-\rightarrow+}^{therm} = \frac{m^4}{4\hbar^3 p^2} e^{h_c} \int_{\sqrt{\frac{2U_{c-}}{m}}}^{\infty} \int_0^{\infty} \mathbf{v}_x \mathbf{v}_{yz} \left(\mathbf{v}_x^2 + \mathbf{v}_{yz}^2\right) e^{-m\left(\mathbf{v}_x^2+\mathbf{v}_{yz}^2\right)/2kT} d\mathbf{v}_{yz} d\mathbf{v}_x$$

let:

$$a = \frac{m}{2kT}$$

then:

$$\mathbf{S}_{n-\rightarrow+}^{therm} = \frac{m^4}{4\hbar^3 p^2} e^{h_c} \int_{\sqrt{\frac{2U_{c-}}{m}}}^{\infty} \left(-\frac{\mathbf{v}_x}{2a^2} - \frac{\mathbf{v}_x\left(\mathbf{v}_x^2+\mathbf{v}_{yz}^2\right)}{2a}\right) e^{-a\left(\mathbf{v}_x^2+\mathbf{v}_{yz}^2\right)} \Bigg|_0^{\infty} d\mathbf{v}_x$$

$$\mathbf{S}_{n-\rightarrow+}^{therm} = \frac{m^3 kT}{4\hbar^3 p^2} e^{h_c} \int_{\sqrt{\frac{2U_{c-}}{m}}}^{\infty} \left(\frac{\mathbf{v}_x}{a} + \mathbf{v}_x^3\right) e^{-a\mathbf{v}_x^2} d\mathbf{v}_x$$

This equation can be converted into the Boltzmann tunneling energy flux.

$$\mathbf{S}_{n-\rightarrow+}^{therm} = \frac{m^3 kT}{4\hbar^3 p^2} e^{h_c} \left(-\frac{1}{a^2} - \frac{\mathbf{v}_x^2}{2a}\right) e^{-a\mathbf{v}_x^2} \Bigg|_{\sqrt{\frac{2U_{c-}}{m}}}^{\infty}$$

$$\mathbf{S}_{n-\rightarrow+}^{therm} = \frac{m(kT)^2}{2\hbar^3 p^2} e^{h_c} \left(2kT + U_{c-}\right) e^{-U_{c-}/kT}$$

$$\mathbf{S}_{n-\rightarrow+}^{therm} = \frac{A_n^* kT^3}{q} \left(2 + \frac{U_{c-}}{kT}\right) e^{h_c - U_{c-}/kT}$$

12.4 Derivation 4 - Fermi-Dirac Thermionic Emission Energy Flux

$$\mathbf{S}_{n-\rightarrow+}^{therm} = \int_{U_{c-}}^{\infty} \mathbf{v}_x (E - E_c) g(E) f(E) dE$$

let:

$$h_c = (E_{fn} - E_c)/kT \qquad E - E_c = \tfrac{1}{2} m\mathbf{v}^2 \qquad dE = m\mathbf{v}d\mathbf{v}$$

then:

$$\mathbf{S}_{n-\to+}^{therm} = \tfrac{1}{2p^2}\left(\tfrac{2m}{\hbar^2}\right)^{\frac{3}{2}} \int_{U_{c-}}^{\infty} \frac{\mathbf{v}_x (E-E_c)^{\frac{3}{2}}}{1+e^{(E-E_c)/kT-h_c}}\,dE$$

$$\mathbf{S}_{n-\to+}^{therm} = \tfrac{1}{2p^2}\left(\tfrac{2m}{\hbar^2}\right)^{\frac{3}{2}}\left(\tfrac{m}{2}\right)^{\frac{3}{2}} m \int_{\sqrt{\frac{2U_{c-}}{m}}}^{\infty} \frac{\mathbf{v}_x \mathbf{v}^4}{1+e^{m\mathbf{v}^2/2kT-h_c}}\,d\mathbf{v}$$

Convert to cylindrical coordinates:

$$\mathbf{S}_{n-\to+}^{therm} = \frac{m^4}{4\hbar^3 p^2} \int_{\sqrt{\frac{2U_{c-}}{m}}}^{\infty}\int_0^{\infty} \frac{\mathbf{v}_x \mathbf{v}_{yz}\left(\mathbf{v}_x^2+\mathbf{v}_{yz}^2\right)}{1+e^{m\left(\mathbf{v}_x^2+\mathbf{v}_{yz}^2\right)/2kT-h_c}}\,d\mathbf{v}_{yz}\,d\mathbf{v}_x$$

let:

$$a = \frac{m}{2kT}$$

then:

$$\mathbf{S}_{n-\to+}^{therm} = \frac{m^4}{4\hbar^3 p^2} \int_{\sqrt{\frac{2U_{c-}}{m}}}^{\infty} \frac{\mathbf{v}_x}{2}\left.\left\{\begin{array}{r}\frac{\left(\mathbf{v}_x^2+\mathbf{v}_{yz}^2\right)^2}{2} - \frac{\left(\mathbf{v}_x^2+\mathbf{v}_{yz}^2\right)\ln\left(1+e^{a\left(\mathbf{v}_x^2+\mathbf{v}_{yz}^2\right)-h_c}\right)}{a} \\ -\frac{1}{a^2}\mathrm{Li}_2\left(-e^{a\left(\mathbf{v}_x^2+\mathbf{v}_{yz}^2\right)-h_c}\right)\end{array}\right\}\right|_0^{\infty} d\mathbf{v}_x$$

$$\mathbf{S}_{n-\to+}^{therm} = \frac{m^4}{4\hbar^3 p^2} \int_{\sqrt{\frac{2U_{c-}}{m}}}^{\infty} \frac{\mathbf{v}_x}{2}\left.\left\{\begin{array}{r}\frac{\left(\mathbf{v}_x^2+\mathbf{v}_{yz}^2\right)^2}{2} - \frac{\left(\mathbf{v}_x^2+\mathbf{v}_{yz}^2\right)\ln\left(1+e^{a\left(\mathbf{v}_x^2+\mathbf{v}_{yz}^2\right)-h_c}\right)}{a} \\ -\frac{1}{a^2}\mathrm{Li}_2\left(-e^{a\left(\mathbf{v}_x^2+\mathbf{v}_{yz}^2\right)-h_c}\right)\end{array}\right\}\right|_0^{\infty} d\mathbf{v}_x$$

Considerable algebra to show that:

$$\mathbf{S}_{n-\to+}^{therm} = \frac{m^2(kT)^2}{4\hbar^3 p^2} \int_{\sqrt{\frac{2U_{c-}}{m}}}^{\infty} \mathbf{v}_x\left\{h_c^2 + \tfrac{p^2}{3} - \ln^2\left(e^{-a\mathbf{v}_x^2}+e^{-h_c}\right) - 2\,\mathrm{Li}_2\left(\tfrac{1}{1+e^{h_c - a\mathbf{v}_x^2}}\right)\right\} d\mathbf{v}_x$$

This equation can be converted into the Fermi-Dirac tunneling energy flux.

$$\mathbf{S}_{n-\to+}^{therm}=\frac{m^2\left(kT\right)^2}{4\hbar^3p^2}\int_{\sqrt{\frac{2U_{c-}}{m}}}^{\infty}\mathbf{v}_x\left\{\begin{matrix}h_c^2+\frac{p^2}{3}-a^2\mathbf{v}_x^4+2a\mathbf{v}_x^2\ln^2\left(1+e^{a\mathbf{v}_x^2-h_c}\right)\\ +2\,\mathrm{Li}_2\left(-e^{a\mathbf{v}_x^2-h_c}\right)\end{matrix}\right\}d\mathbf{v}_x$$

$$\mathbf{S}_{n-\to+}^{therm}=\frac{m^2\left(kT\right)^2}{4\hbar^3p^2}\left\{\left(h_c^2+\frac{p^2}{3}\right)\frac{\mathbf{v}_x^2}{2}-\frac{a^2\mathbf{v}_x^6}{6}-\mathbf{v}_x^2\,\mathrm{Li}_2\left(-e^{a\mathbf{v}_x^2-h_c}\right)+\frac{2}{a}\,\mathrm{Li}_3\left(-e^{a\mathbf{v}_x^2-h_c}\right)\right\}\Bigg|_{\sqrt{\frac{2U_{c-}}{m}}}^{\infty}$$

$$\mathbf{S}_{n-\to+}^{therm}=\frac{m\left(kT\right)^2}{4\hbar^3p^2}\left\{\left(h_c^2+\frac{p^2}{3}\right)E-\frac{E^3}{3\left(kT\right)^2}-2E\,\mathrm{Li}_2\left(-e^{E/kT-h_c}\right)+4kT\,\mathrm{Li}_3\left(-e^{E/kT-h_c}\right)\right\}\Bigg|_{U_{c-}}^{\infty}$$

Large amount of algebra to show that this equation evaluated at infinity is:

$$\frac{2}{3}kTh_c\left(h_c^2+p^2\right)$$

Even more algebra to show that:

$$\mathbf{S}_{n-\to+}^{therm}=\frac{A_n^*kT^3}{q}\left[\begin{matrix}2\left\{\mathrm{Li}_3\left(\frac{1}{1+e^{U_{c-}/kT-h_c}}\right)+\mathrm{Li}_3\left(\frac{1}{1+e^{h_c-U_{c-}/kT}}\right)-\mathrm{Li}_3(1)\right\}\\ +\left(\frac{U_{c-}}{kT}\right)\mathrm{Li}_2\left(\frac{1}{1+e^{U_{c-}/kT-h_c}}\right)\\ +\frac{1}{3}h_c\left(h_c^2+p^2\right)-\left(\frac{U_{c-}}{kT}\right)\left(\frac{h_c^2}{2}+\frac{p^2}{3}\right)+\frac{1}{6}\left(\frac{U_{c-}}{kT}\right)^3\\ -\ln\left(1+e^{U_{c-}/kT-h_c}\right)\left\{\begin{matrix}\frac{2}{3}\ln^2\left(1+e^{U_{c-}/kT-h_c}\right)\\ -\left(\frac{3}{2}\frac{U_{c-}}{kT}-h_c\right)\ln\left(1+e^{U_{c-}/kT-h_c}\right)\\ +\left(\frac{U_{c-}}{kT}\left(\frac{U_{c-}}{kT}-h_c\right)-\frac{p^2}{3}\right)\end{matrix}\right\}\end{matrix}\right]$$