13. Appendix H - Special Functions

13.1 Incomplete Gamma Function

$$\gamma(a,x) = \int_0^x e^{-t} t^{a-1} dt = x^a \sum_{n=0}^{\infty} \frac{(-x)^n}{(a+n)n!}$$

13.2 Gamma Function

$$\Gamma(j) = \gamma(j,\infty)$$

$$\Gamma(j+1) = j\Gamma(j) \qquad \Gamma(1) = 1 \qquad \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

13.3 Fermi-Dirac Distribution Function

$$f(x,\eta) = \frac{1}{1+e^{x-\eta}}$$

13.4 Fermi Integral

$$\mathcal{F}_j(\eta) = \int_0^{\infty} x^j f(x,\eta) dx$$ For $x >> \eta$: $\mathcal{F}_j(\eta) = \Gamma(j+1)e^{\eta}$

$$F_j(\eta) = \mathcal{F}_j(\eta) / \Gamma(j+1)$$ For $x >> \eta$: $F_j(\eta) = e^{\eta}$

$$\frac{dF_j(\eta)}{d\eta} = F_{j-1}(\eta)$$

$$\int_0^{\infty} x^j \frac{\partial f(x)}{\partial x} dx = -j\Gamma(j) F_{j-1}(\eta)$$ $j > 0$

$$F_0(\eta) = \ln(1+e^{\eta}) \qquad F_{-1}(\eta) = \frac{1}{1+e^{-\eta}}$$

$$\frac{F_j(\eta)}{F_k(\eta)} = 1$$ For $x >> \eta$ and any j,k

Numerical Approximation - See Reference [65]

$$\mathcal{F}_j(\eta) \approx \left(\frac{(j+1)2^{j+1}}{\left[b+\eta+\left(|\eta-b|^c + a^c\right)^{1/c}\right]^{j+1}} + \frac{e^{-\eta}}{\Gamma(j+1)} \right)^{-1}$$

$$a = \left[1 + \tfrac{15}{4}(j+1) + \tfrac{1}{40}(j+1)^2\right]^{0.5}$$

$$b = 1.8 + 0.61j$$

$$c = 2 + (2-\sqrt{2})2^{-j}$$

13.5 Dilogarithm

See Reference [66,67]

$$\mathrm{Li}_2(x) = -\int_0^x \frac{\ln(1-z)}{z}dz = \sum_{k=1}^{\infty}\frac{x^k}{k^2} \qquad \text{when } |x| \le 1$$

$$\mathrm{Li}_2(1) = \tfrac{p^2}{6}$$

$$\mathrm{Li}_2(-x) = \mathrm{Li}_2\left(\tfrac{1}{1+x}\right) - \tfrac{p^2}{6} + \tfrac{1}{2}\ln(1+x)\ln\left(\tfrac{1+x}{x^2}\right) \qquad \text{when } x > 0$$

$$\mathrm{Li}_2(x) = -\mathrm{Li}_2(1-x) + \tfrac{p^2}{6} - \ln(x)\ln(1-x)$$

13.6 Trilogarithm

See Reference [66,67]

$$\mathrm{Li}_3(x) = -\int_0^x \frac{\mathrm{Li}_2(z)}{z}dz = \sum_{k=1}^{\infty}\frac{x^k}{k^3} \qquad \text{when } |x| \le 1$$

$$\mathrm{Li}_3(1) = 1.20205690$$

$$\begin{aligned}\mathrm{Li}_3(-x) = &-\mathrm{Li}_3(\tfrac{x}{1+x}) - \mathrm{Li}_3(\tfrac{1}{1+x}) + \mathrm{Li}_3(1) \\ &-\tfrac{p^2}{6}\ln(1+x) - \tfrac{1}{2}\ln(x)\ln^2(1+x) + \tfrac{1}{3}\ln^3(1+x)\end{aligned} \qquad \text{when } x > 0$$

13.7 Bernoulli Function

$$B(x) = \frac{x}{e^x - 1} \qquad \text{for small } x\text{: } B(x) \approx \frac{1}{1 + \frac{1}{2}x + \frac{1}{6}x^2}$$

$$\frac{dB(x)}{dx} = \frac{xe^x - e^x - 1}{(e^x - 1)^2} \qquad \text{for small } x\text{: } \frac{dB(x)}{dx} \approx \frac{-\frac{1}{2} - \frac{1}{3}x - \frac{1}{8}x^2}{1 + x + \frac{7}{12}x^2}$$