

Green's Functions

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1 The basic idea

Consider a general non-homogeneous equation in a domain Ω

$$\mathbf{L}_y u(\mathbf{y}) = f(\mathbf{y}), \quad \mathbf{y} \in \Omega, \quad (1.1)$$

where \mathbf{L}_y is a linear operator and f is given; the unknown function u is supposed to satisfy some boundary condition on the boundary $\partial\Omega$ of Ω :

$$\mathbf{B}u = b(\mathbf{y}) \text{ for } \mathbf{y} \in \partial\Omega, \quad (1.2)$$

where \mathbf{B} is a suitable operator (for instance, the identity for Dirichlet data, or the normal gradient for Neumann data). We assume that u and f belong to a suitable Hilbert space in which a scalar product is defined in terms of an integral over the domain Ω .

Alongside this problem, consider the related problem

$$\mathbf{L}_y^+ G(\mathbf{y}; \mathbf{x}) = \delta(\mathbf{y} - \mathbf{x}), \quad (1.3)$$

where the operator \mathbf{L}_y^+ also acts on the variable \mathbf{y} and will be specified presently. Since this is an equation in which the variable is \mathbf{y} , G depends on the point \mathbf{x} – which is to be considered fixed – only as a parameter. Take the scalar product of (1.1) with G

$$(G, \mathbf{L}_y u) = (G, f), \quad (1.4)$$

and suppose that the operators \mathbf{L} and \mathbf{L}^+ are such that, for any two functions u and v (belonging in general to a suitably restricted class), one can write

$$(v, \mathbf{L}_y u) = (\mathbf{L}_y^+ v, u) + J[v, u]. \quad (1.5)$$

The expression $J[u, v]$ is called the *conjunct* of u and v and depends on the values of these functions on the boundary $\partial\Omega$ of Ω . Upon using this relation in (1.4) and the equation satisfied by G , (1.3), we find

$$\langle \delta, u \rangle = (G, f) - J[G, u]. \quad (1.6)$$

By the sifting action of the δ , the first term can be evaluated giving $u(\mathbf{x})$ ¹ and we thus have

$$u(\mathbf{x}) = (G, f) - J[G, u], \quad (1.7)$$

where, it will be noted, both (G, f) and $J[G, u]$ are functions of \mathbf{x} . The conjunct J involves u on the boundary and therefore, given (1.2), it may or may not be completely known. For example, it might involve both u itself as well as its normal gradient on the boundary, and it may be that only one or the other is known, but not both. In this case, Eq. (1.7) is only a formal relation as the unknown appears on both sides of the equation. But suppose that one can impose boundary conditions on G which eliminate the unknown information about u : in this case (1.7) is the complete, explicit solution of the problem. A function G determined in this way is called the *Green's function* of the problem (1.1), (1.2). Physically, it frequently represents the response of the system to a point source or a unit impulse. For this reason, depending on the physical context, it is sometimes referred to as the *fundamental solution* or the *impulse response*.

If we substitute (1.7) into the original equation (1.1), we would write

$$\mathbf{L}_x u(\mathbf{x}) = \mathbf{L}_x (G, f) - \mathbf{L}_x J[G, u] = (\mathbf{L}_x G, f) - J[\mathbf{L}_x G, u] = f(\mathbf{x}). \quad (1.8)$$

For this equation to be satisfied it is evidently necessary that

$$\mathbf{L}_x G(\mathbf{y}; \mathbf{x}) = \delta(\mathbf{x} - \mathbf{y}) \quad (1.9)$$

Since, by construction, the conjunct is evaluated for points \mathbf{y} on the boundary, it is clear that, if this relation is satisfied, $\mathbf{L}_x G$ vanishes whenever \mathbf{x} is not on the boundary so that $(\mathbf{L}_x G, f) = 0$. We thus see that the Green's function satisfies the adjoint equation in the first variable and the original equation in the second one. The situation in which both \mathbf{x} and \mathbf{y} are one the boundary is a special one and needs to be addressed specifically case by case.

¹If the scalar product is defined with a weight function w , we might get $w(\mathbf{x}) u(\mathbf{x})$ in place of $u(\mathbf{x})$ depending on the form of the δ .

2 Poisson Equation

Consider the Poisson equation in three-dimensional space which, following convention, we write as²

$$\nabla_y^2 u(\mathbf{y}) = -4\pi f(\mathbf{y}). \quad (2.1)$$

In order to find the Green's function we first prove *Green's identity*.

Let u and v be two sufficiently smooth functions defined in a volume V and on the surface S of V . The surface can consist of several disconnected pieces, which are collectively denoted by S . If the volume extends to infinity, we suppose that u and v are well-behaved at infinity. It is trivial to prove that

$$v \nabla_y^2 u - u \nabla_y^2 v = \nabla \cdot (v \nabla_y u - u \nabla_y v). \quad (2.2)$$

Upon integrating over the volume V and using the divergence theorem we then have

$$\int d^3 y (v \nabla_y^2 u - u \nabla_y^2 v) = \int_S dS_y (v \nabla_y u - u \nabla_y v) \cdot \mathbf{n}_y, \quad (2.3)$$

where \mathbf{n}_y is the unit normal directed outward with respect to V . If S consists of disconnected pieces, the surface integral is to be understood as a sum over each piece. If V is the domain exterior to some surface and extending to infinity, S in this formula includes a surface at infinity; in most cases of interest the integral over this surface vanishes. From Eq. (2.3) we read L^+ and J :

$$L_y^+ v = \nabla_y^2 v, \quad J[v, u] = \int_S dS_y (v \nabla_y u - u \nabla_y v) \cdot \mathbf{n}_y. \quad (2.4)$$

Now we write (1.3) in a slightly different form, motivated by (2.1):

$$\nabla_y^2 G(\mathbf{y}; \mathbf{x}) = -4\pi \delta(\mathbf{y} - \mathbf{x}). \quad (2.5)$$

By suitably modifying (1.7) to account for the different signs and the extra factor of 4π we find

$$\int_V d^3 y f(\mathbf{y}) G(\mathbf{y}; \mathbf{x}) + \frac{1}{4\pi} \int_S dS_y (G \nabla_y u - u \nabla_y G) \cdot \mathbf{n}_y = \begin{cases} u(\mathbf{x}) & \text{if } \mathbf{x} \in V \\ 0 & \text{if } \mathbf{x} \notin V \end{cases}. \quad (2.6)$$

If the mathematical nature of the problem (2.1) required the specification of both u and $\mathbf{n} \cdot \nabla u$ on S , the formula (2.6) would be a closed-form solution of the problem only expressed in terms of the data. We shall see that this is actually not possible, as the boundary conditions on S can only involve *either* u (and in that case we refer to **Dirichlet** boundary conditions) or $\mathbf{n} \cdot \nabla u$ (**Neumann** conditions). Because of this circumstance, (2.6) in general is only an integral statement.

Remarks

1. The equation for G is linear and non-homogeneous and, therefore, its general solution is the sum of any one particular solution and the general solution of the homogeneous equation. Since in three dimensions

$$\nabla^2 \frac{1}{|\mathbf{x} - \mathbf{y}|} = -4\pi \delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad (2.7)$$

a particular solution is

$$G_{part} = \frac{1}{|\mathbf{x} - \mathbf{y}|}, \quad (2.8)$$

²In electrostatics, the field \mathbf{E} is minus the gradient of the potential and, if CGS units are used, Gauss's law is written as $\nabla \cdot \mathbf{E} = 4\pi\rho$, where ρ is the charge density; in these units the potential of a point charge is q/r .

which, when substituted into (2.6), gives

$$\int_V d^3y \frac{f(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} + \frac{1}{4\pi} \int_S dS_y \left[\frac{1}{|\mathbf{x} - \mathbf{y}|} \nabla_y u - u \nabla_y \left(\frac{1}{|\mathbf{x} - \mathbf{y}|} \right) \right] \cdot \mathbf{n}_y = \begin{cases} u(\mathbf{x}) & \text{if } \mathbf{x} \in V \\ 0 & \text{if } \mathbf{x} \notin V \end{cases} \quad (2.9)$$

This relation is important, among other things, as the starting point for the *boundary integral method*, a powerful method for the numerical solution of the Laplace equation. In electrostatics, G_{part} is of course the potential generated by a unit point charge located at \mathbf{x} .

Returning to the general problem, in view of the linearity of the equation for G , we may write the most general solution as

$$G(\mathbf{y}; \mathbf{x}) = \frac{1}{|\mathbf{x} - \mathbf{y}|} + F(\mathbf{y}; \mathbf{x}), \quad (2.10)$$

where $\nabla_y^2 F = 0$.

2. In particular, if the volume V is the entire space and $u \rightarrow 0$ at infinity, $F = 0$ and, provided u decreases faster than $1/|\mathbf{x}|$, the surface integral vanishes and (2.6) reduces to

$$u(\mathbf{x}) = \int \frac{f(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} d^3y, \quad (2.11)$$

which is a familiar result, e.g. from electrostatics. In this case, for example, the physical interpretation of this result is obvious: given a charge distributed through space with a density f , each element of volume can be considered a point charge of magnitude $f d^3y$, and (2.11) is just the superposition of the effects of all these elementary volumes.

3. The surface integral in (2.6) is a solution of $\nabla^2 u = 0$ at all points $\mathbf{x} \notin S$, since $\nabla^2 |\mathbf{x} - \mathbf{y}|^{-1} = 0$ at these points. Hence the only part related to the non-vanishing right-hand side of Poisson's equation is the volume integral. Earlier we have already considered this equation and we have obtained a particular solution in the form

$$u = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} R_{\ell m} Y_{\ell}^m \quad (2.12)$$

where

$$R_{\ell m} = R_{\ell m}^{homog} + \frac{r^{\ell}}{2\ell + 1} \int_c^r f_{\ell m} r^{-\ell+1} dr - \frac{r^{-\ell-1}}{2\ell + 1} \int_{c'}^r f_{\ell m} r^{\ell+2} dr \quad (2.13)$$

with $f_{\ell m} = (Y_{\ell}^m, f)$. It can be shown by taking a scalar product of (2.11) with Y_{ℓ}^m , using the generating function for the Legendre polynomials, and the addition theorem for the spherical harmonics, that the particular solution corresponds to the volume integral (2.11) (up to factors of 4π , etc.).

4. In two space dimensions one has

$$\nabla^2 \log |\mathbf{x} - \mathbf{y}| = -2\pi \delta^{(2)}(\mathbf{x} - \mathbf{y}), \quad (2.14)$$

and the equation for G is written as

$$\nabla^2 G = -2\pi \delta^{(2)}(\mathbf{x} - \mathbf{y}). \quad (2.15)$$

Hence the particular solution analogous to (2.8) is

$$G_{part} = \log |\mathbf{x} - \mathbf{y}|. \quad (2.16)$$

2.1 Dirichlet problem

In the Dirichlet problem u is prescribed on the boundary S . Suppose we can find a solution of the equation for G such that $G(\mathbf{y}; \mathbf{x}) = 0$ for $\mathbf{y} \in S$. If G is decomposed as in (2.10), the function F must then satisfy the condition

$$F(\mathbf{y}; \mathbf{x}) = -\frac{1}{|\mathbf{x} - \mathbf{y}|} \quad \mathbf{y} \in S. \quad (2.17)$$

With this choice we then we have

$$u(\mathbf{x}) = \int f(\mathbf{y}) G(\mathbf{y}; \mathbf{x}) d^3y - \frac{1}{4\pi} \int_S u(\mathbf{y}) \frac{\partial}{\partial n_y} G(\mathbf{y}; \mathbf{x}) dS_y \quad (2.18)$$

which represents an explicit solution to the problem. In this way, solving the problem is reduced to the calculation of the appropriate G , dependent only on the geometry, after which *any* Dirichlet problem for the Poisson equation with the given geometry can be solved by the previous formula for any boundary data. In a formal sense, we may rewrite (2.18) as

$$u(\mathbf{y}) = (\nabla_y^{-2})_D f(\mathbf{y}), \quad (2.19)$$

where $(\nabla_y^{-2})_D$ is the *inverse* of the operator ∇_y^2 for the Dirichlet problem. We will see below in Eq. (2.34) a formula for the Neumann problem which is similar in essence, but different in detail in view of the difference in the information available on the boundary. This point illustrates that the complete definition of an *operator*, in addition to the *operations* associated with the operator, must include a specification of the boundary conditions.

We can show that the Green's function G for the Poisson equation with Dirichlet boundary conditions is symmetric, i.e.

$$G(\mathbf{x}; \mathbf{y}) = G(\mathbf{y}; \mathbf{x}). \quad (2.20)$$

To this end let

$$U(\mathbf{z}) = G(\mathbf{z}; \mathbf{x}), \quad V(\mathbf{z}) = G(\mathbf{z}; \mathbf{y}). \quad (2.21)$$

Then, from (2.5),

$$\nabla_z^2 U = -4\pi\delta^{(3)}(\mathbf{z} - \mathbf{x}), \quad \nabla_z^2 V = -4\pi\delta^{(3)}(\mathbf{z} - \mathbf{y}). \quad (2.22)$$

Consider now Green's identity (2.3) written for U and V . With (2.22), the left-hand side can be calculated to be

$$4\pi [V(\mathbf{x}) - U(\mathbf{y})] = \int_S (U\nabla_z V - V\nabla_z U) \cdot \mathbf{n}_z dS_z. \quad (2.23)$$

But, given that G has been so constructed as to vanish when its first argument $\mathbf{z} \in S$, the right-hand side vanishes and we are left with

$$V(\mathbf{x}) - U(\mathbf{y}) = 0, \quad (2.24)$$

which proves (2.20). Because of this symmetry, from now on we will not need to treat differently the variables \mathbf{x} and \mathbf{y} .

Physically (2.20) implies a *principle of reciprocity* between source point and field point: the effect at \mathbf{x} (field point) of a point charge at \mathbf{y} (source point) is the same as that at \mathbf{y} of a point charge at \mathbf{x} .

Example. As an application let us consider the problem outside a spherical boundary of radius a on which u is given, $u = U(\mathbf{x})$. To calculate the Green's function we need to solve

$$\nabla_x^2 G = \nabla_x^2 v = -4\pi\delta^{(3)}(\mathbf{x} - \mathbf{x}'), \quad G(\mathbf{x}; \mathbf{x}') = 0 \quad \mathbf{x} \in S. \quad (2.25)$$

Since we know that the result will be symmetric in \mathbf{x} and \mathbf{x}' , we do not need to worry about whether the operator acts on one or the other argument. In one of the examples given for the Laplace equation in

spherical geometry we have already solved this problem when $G = u_0$ on the sphere surface and from that expression, setting $u_0 = 0$, we deduce

$$G = \frac{1}{|\mathbf{x} - \mathbf{x}'|} - \frac{a/|\mathbf{x}'|}{|\mathbf{x} - \mathbf{x}'_I|} \quad \text{where} \quad \mathbf{x}'_I = \frac{a^2}{|\mathbf{x}'|} \frac{\mathbf{x}'}{|\mathbf{x}'|} \quad (2.26)$$

$$= \frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos \gamma}} - \frac{1}{\sqrt{r^2 r'^2 / a^2 + a^2 - 2rr' \cos \gamma}} \quad (2.27)$$

where $\cos \gamma = \mathbf{x} \cdot \mathbf{x}' / |\mathbf{x}| |\mathbf{x}'|$. Clearly $G(\mathbf{x}; \mathbf{x}') = G(\mathbf{x}'; \mathbf{x})$, and $G = 0$ on the sphere. To apply Green's formula, we need $\partial G / \partial n'$:

$$\frac{\partial G}{\partial n'} = -\frac{\partial G}{\partial r} = \frac{r' - r \cos \gamma}{[\sqrt{r^2 + r'^2 - 2rr' \cos \gamma}]^3} - \frac{(r^2/a^2)r' - r \cos \gamma}{[\sqrt{r^2 r'^2 / a^2 + a^2 - 2rr' \cos \gamma}]^3} \quad (2.28)$$

where the minus sign arises from the fact that the normal is directed out of the domain of interest and, therefore, into the sphere. Upon evaluating this expression on the sphere, we find

$$\left. \frac{\partial G}{\partial n'} \right|_{r'=a} = \frac{a^2 - r^2}{a [\sqrt{r^2 + a^2 - 2ra \cos \gamma}]^3} \quad (2.29)$$

Therefore

$$u = \int d^3x' f(\mathbf{x}') G(\mathbf{x}; \mathbf{x}') - \frac{a(a^2 - r^2)}{4\pi} \int_{\Omega} \frac{U(\theta', u')}{[\sqrt{r^2 + a^2 - 2ar \cos \gamma}]^3} d\Omega', \quad (2.30)$$

where $\cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi')$. For the Laplace equation $f = 0$ and this reduces to Poisson's exterior formula. For the interior problem $\partial / \partial n' = \partial / \partial r$, the minus becomes a plus, and we recover an earlier result.

2.2 Neumann problem

For the Neumann problem it would seem that the natural choice for the boundary condition on G would be $\partial G / \partial n' = 0$ on the boundary. However, integrating (2.5) over the volume and using the divergence theorem we find that

$$\int_S dS' \mathbf{n}' \cdot \nabla' G = -4\pi, \quad (2.31)$$

which would be violated if the normal derivative were zero. The simplest condition that would respect this constraint is

$$\mathbf{n} \cdot \nabla G = -\frac{4\pi}{S} \quad (2.32)$$

on the boundary. Upon noting that

$$\frac{1}{S} \int dS' u(\mathbf{x}') = \langle u \rangle_S \quad (2.33)$$

is the average of u over the boundary, Green's formula becomes

$$u(\mathbf{x}) - \langle u \rangle_S = \int dV' f(\mathbf{x}') G(\mathbf{x}'; \mathbf{x}) + \frac{1}{4\pi} \int_S dS' G(\mathbf{x}'; \mathbf{x}) \mathbf{n}' \cdot \nabla' u \quad (2.34)$$

The constant $\langle u \rangle_S$ is undetermined, but this lack of uniqueness is clearly inherent in the nature of the Neumann problem where information is only given on derivatives of u . Of course, for the exterior problem, if it is required that $u \rightarrow 0$ at infinity (which is really a Dirichlet condition), $\langle u \rangle_S = 0$.

For the Neumann problem the symmetry of G is not automatic but can be imposed.

3 The diffusion equation

Consider the non-homogeneous diffusion equation

$$\frac{\partial u}{\partial \tau} - \nabla_y^2 u = f(\mathbf{y}, \tau) \quad (3.1)$$

in a volume V and over the time interval $0 < \tau < t$ subject to boundary conditions that will be specified later. In this case the domain Ω has a space-time nature including both V and $0 < \tau < t$ and we write $\Omega = V \times (0, t)$.

Multiply (3.1) by a function $G(\mathbf{y}, \tau; \mathbf{x}, t)$, and integrate over τ from 0 to t and over V in \mathbf{y} :

$$\int_0^t d\tau \int_V dV_y G \left(\frac{\partial u}{\partial \tau} - \nabla_y^2 u \right) = \int_0^t d\tau \int_V dV_y G f(\mathbf{y}, \tau) \quad (3.2)$$

Some simple manipulations give

$$\begin{aligned} \int_0^t d\tau \int_V dV_y \left(-\frac{\partial G}{\partial \tau} - \nabla_y^2 G \right) u(\mathbf{y}, \tau) &= \int_V dV_y [G(\mathbf{y}, 0; \mathbf{x}, t) u(\mathbf{y}, 0) - G(\mathbf{y}, t; \mathbf{x}, t) u(\mathbf{y}, t)] \\ &+ \int_0^t d\tau \int_S dS_y \mathbf{n} \cdot (G \nabla_y u - u \nabla_y G) + \int_0^t d\tau \int_V dV_y G f(\mathbf{y}, \tau). \end{aligned} \quad (3.3)$$

Choose now G such that

$$-\frac{\partial G}{\partial \tau} - \nabla_y^2 G = \delta^{(3)}(\mathbf{x} - \mathbf{y}) \delta(t - \tau). \quad (3.4)$$

In order to use this relation to reduce the first term to $u(\mathbf{x}, t)$, we need to understand the integration over τ in the sense

$$\int_0^t d\tau (\dots) = \lim_{\epsilon \rightarrow 0} \int_0^{t+\epsilon} d\tau (\dots) \quad (3.5)$$

When appropriate, we will write $t+0$ in place of t to avoid confusion. Equation (3.4) shows that G has a simple discontinuity across $\tau = t$, but no worse singularity (such as, e.g., a δ -like behavior).

In most problems, one is concerned with an initial value problem in which data are prescribed at $\tau = 0$, but not at the final instant $\tau = t$. This circumstance suggests to impose

$$G(\mathbf{y}, t+0; \mathbf{x}, t) = 0, \quad (3.6)$$

Thus, we need to solve (3.4) backward in time.³ Equation (3.3) becomes

$$\begin{aligned} u(\mathbf{x}, t) &= \int_V dV_y G(\mathbf{y}, 0; \mathbf{x}, t) u(\mathbf{y}, 0) + \int_0^{t+0} d\tau \int_S dS_y \mathbf{n} \cdot (G \nabla_y u - u \nabla_y G) \\ &+ \int_0^{t+0} d\tau \int_V dV_y G(\mathbf{y}, \tau; \mathbf{x}, t) f(\mathbf{y}, \tau), \end{aligned} \quad (3.7)$$

which may be called the **generalized Duhamel formula**. This formula is valid in any number of space dimensions. In one dimension, if the interval of interest extends for $a < y < b$, the surface integral in this formula must be interpreted as

$$\int_0^{t+0} d\tau \int_S dS_y \mathbf{n} \cdot (G \nabla_y u - u \nabla_y G) = \int_0^{t+0} d\tau \left[G \frac{\partial u}{\partial y} - u \frac{\partial G}{\partial y} \right]_{y=a}^{y=b}. \quad (3.8)$$

³For normal parabolic equations this causes stability problems, but the equal signs of the two terms in the left-hand side of the diffusion equation (3.4) indicate that the diffusivity parameter is negative in this case (equal to -1 with our normalizations). This is equivalent to solving a diffusion equation with positive diffusivity forward in time and no difficulties arise.

To justify this interpretation one may think of a region in the form of a slab: the unit normal at $y = b$ is the unit vector in the positive y -direction, while that at $y = a$ is the unit vector in the negative y -direction.

In order to make the definition of G unique, we need to specify boundary conditions; this can be done so as to turn (3.3) into an explicit result.

Noting the difference in the sign of the time derivative between (3.4) and (3.1), one may say that G is to be found by integrating backward in time the effect of a ‘puff’ of diffusing entity (be that heat, a solute, or other) released at \mathbf{x} at time t .

3.1 Dirichlet problem

Suppose that

$$u(\mathbf{y}, \tau) = F(\mathbf{y}, \tau) \quad \text{on boundary,} \quad u(\mathbf{y}, 0) = 0. \quad (3.9)$$

Choose then

$$G(\mathbf{y}, \tau; \mathbf{x}, t) = 0 \quad \text{for} \quad \mathbf{y} \in S \quad (3.10)$$

Duhamel’s formula (3.7) reduces to

$$u(\mathbf{x}, t) = - \int_0^{t+0} d\tau \int_S dS_y F \mathbf{n} \cdot \nabla_y G + \int_0^{t+0} d\tau \int_V dV_y G(\mathbf{y}, \tau; \mathbf{x}, t) f(\mathbf{y}, \tau). \quad (3.11)$$

Example. Consider the one-dimensional diffusion problem

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial y^2}, \quad (3.12)$$

in $0 < y < \pi$, subject to the conditions

$$u(0, \tau) = F(\tau), \quad u(\pi, \tau) = 0, \quad u(y, 0) = 0. \quad (3.13)$$

The boundary condition (3.10) suggests to solve for G by using a Fourier sine series:

$$G(y, \tau; x, t) = \sum_{n=1}^{\infty} g_n(\tau; x, t) \sin ny. \quad (3.14)$$

Upon proceeding in the usual way we are led to

$$\frac{\partial g_n}{\partial \tau} - n^2 g_n = -\frac{2}{\pi} \sin nx \delta(\tau - t) \quad (3.15)$$

which may be rewritten as

$$\begin{aligned} \frac{\partial}{\partial \tau} [\exp(-n^2 \tau) g_n] &= -\frac{2}{\pi} \sin nx \exp(-n^2 \tau) \delta(\tau - t) \\ &= -\frac{2}{\pi} \sin nx \exp(-n^2 t) \delta(\tau - t) \end{aligned} \quad (3.16)$$

Since we have a condition at $\tau = t + 0$, it is convenient to integrate from τ to $t + 0$:

$$\exp(-n^2 t) g_n(t + 0; x, t) - \exp(-n^2 \tau) g_n(\tau; x, t) = -\frac{2}{\pi} \sin nx \exp(-n^2 t) \int_{\tau}^{t+0} \delta(\sigma - t) d\sigma \quad (3.17)$$

where we have changed the name of the integration variable to avoid confusion with the starting point τ . The first term vanishes due to (3.6) and the integral of the δ equals 1 only if $\tau < t$, which can be ensured by writing the result of the integral as $H(t - \tau)$.⁴ Hence we find

$$g_n(\tau; x, t) = \frac{2}{\pi} \exp[-n^2(t - \tau)] \sin nx H(t - \tau) \quad (3.18)$$

so that

$$G = \frac{2}{\pi} H(t - \tau) \sum_{n=1}^{\infty} \sin nx \sin ny e^{-n^2(t-\tau)}. \quad (3.19)$$

When this expression is substituted into (3.11) and use is made of (3.8), the result is

$$u(x, t) = \int_0^t d\tau F(\tau) \left[\frac{2}{\pi} \sum_{n=1}^{\infty} n \sin nx e^{-n^2(t-\tau)} \right], \quad (3.20)$$

where we have dropped the H function since the integral is from 0 to t anyway. This is the same result as we found by using the Fourier series to solve (3.12) directly.

3.2 Neumann problem

If now

$$\mathbf{n} \cdot \nabla u(\mathbf{y}, \tau) = H(\mathbf{y}, \tau) \quad \text{on boundary}, \quad u(\mathbf{y}, 0) = 0, \quad (3.21)$$

we choose

$$\mathbf{n}_y \cdot \nabla_y G(\mathbf{y}, \tau; \mathbf{x}, t) = 0 \quad \text{for} \quad \mathbf{y} \in S, \quad (3.22)$$

to find

$$u(\mathbf{x}, t) = \int_0^t d\tau \int_S dS_y G(\mathbf{y}, \tau; \mathbf{x}, t) H(\mathbf{y}, \tau) + \int_0^t d\tau \int_V dV_y G(\mathbf{y}, \tau; \mathbf{x}, t) f(\mathbf{y}, \tau), \quad (3.23)$$

3.3 Initial-value problem

Let

$$u(\mathbf{y}, \tau) = 0 \quad \text{on boundary}, \quad u(\mathbf{y}, 0) = u_0(\mathbf{y}). \quad (3.24)$$

We take

$$G(\mathbf{y}, \tau; \mathbf{x}, t) = 0 \quad \text{for} \quad \mathbf{y} \in S \quad (3.25)$$

and

$$u(\mathbf{x}, t) = \int_V dV_y G(\mathbf{y}, 0; \mathbf{x}, t) u_0(\mathbf{y}) + \int_0^t d\tau \int_V dV_y G(\mathbf{y}, \tau; \mathbf{x}, t) f(\mathbf{y}, \tau), \quad (3.26)$$

Similarly, if

$$\mathbf{n} \cdot \nabla_y u(\mathbf{y}, \tau) = 0 \quad \text{on boundary}, \quad u(\mathbf{y}, 0) = v_0(\mathbf{y}), \quad (3.27)$$

we choose

$$\mathbf{n}_y \cdot \nabla_y G(\mathbf{y}, \tau; \mathbf{x}, t) = 0 \quad \text{for} \quad \mathbf{y} \in S \quad (3.28)$$

and

$$u(\mathbf{x}, t) = \int_V dV_y G(\mathbf{y}, 0; \mathbf{x}, t) v_0(\mathbf{y}) + \int_0^t d\tau \int_V dV_y G(\mathbf{y}, \tau; \mathbf{x}, t) f(\mathbf{y}, \tau), \quad (3.29)$$

⁴Note that, as implied by our notation $t + 0$, in principle the variable τ ranges between 0 and an arbitrarily small amount past t .

Example. Consider the one dimensional problem (3.12) on the infinite line $-\infty < y < \infty$, with conditions $u \rightarrow 0$ at infinity and

$$u(y, 0) = F(y). \quad (3.30)$$

The solution of (3.4) decaying at infinity can be found e.g. by means of the Fourier transform and is

$$G(y, \tau; x, t) = \frac{1}{2\sqrt{\pi(t-\tau)}} \exp\left[-\frac{(x-y)^2}{4(t-\tau)}\right]. \quad (3.31)$$

The solution of the problem takes then the well-known form

$$u(x, t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} dy F(y) \exp\left[-\frac{(x-y)^2}{4t}\right]. \quad (3.32)$$

4 Sturm-Liouville Problems

Let us now consider a problem of the form (1.1) in which the operator \mathbf{L} is an ordinary differential operator on the interval $a < x < b$ with the form

$$\mathbf{L}u = -\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + q(x)u. \quad (4.1)$$

The given functions $p(x)$ and $q(x)$ are real, p, p' , and q are continuous. We assume that p does not vanish or change sign in $[a, b]$ and therefore, without loss of generality, we may assume and $p > 0$. If $[a, b]$ is finite and the assumptions on the coefficients hold in the closed interval (i.e., including the end-points), the problem is said to be *regular*, otherwise it is *singular*.⁵

To completely specify the operator, we need to give the boundary conditions which we take to be of the fairly general type

$$a_{11}u(a) + a_{12}u'(a) = \gamma_1, \quad a_{21}u(b) + a_{22}u'(b) = \gamma_2 \quad (4.2)$$

where the a_{ij} are real constants. Note that, while general, this class of boundary condition is not the most general one as it does not “mix” information at the two endpoints. For example, we are excluding a condition of the type $u(a) + Bu'(b) = \gamma$. This we do for simplicity.

⁵The standard Sturm-Liouville form (4.1) is less restrictive than it might seem. Indeed, consider the general second-order linear differential operator

$$\mathbf{D}u \equiv F(x)u'' + G(x)u' + Hu$$

Let us look for a multiplicative factor ϕ such that

$$\phi \mathbf{D}u = -(pu')' + qu,$$

for some p and q . Clearly we must have

$$\phi F = -p \quad \phi G = -p'.$$

Take the ratio

$$\frac{p'}{p} = \frac{G}{F} \quad \Rightarrow \quad p = A \exp\left(\int \frac{G}{F} dx\right)$$

which will be positive if we take the integration constant $A > 0$. Furthermore, we must have

$$\phi = -\frac{p}{F}, \quad q = \phi H = -\frac{pH}{F}.$$

Hence, given the problem

$$\mathbf{D}u = f,$$

we can transform it to

$$\phi \mathbf{D}u \equiv -(pu')' + qu = \phi f,$$

which has the standard Sturm-Liouville structure (4.1).

To derive the analog of (1.5) we multiply the left-hand side of (4.1) by v and integrate by parts to find

$$\int_a^b v [-(pu')' + qu] dx = \int_a^b [-(pv')' + qv] u dx + [p(v'u - vu')]_a^b, \quad (4.3)$$

from which we see that $\mathbb{L}^+ = \mathbb{L}$ and

$$J = [p(v'u - vu')]_a^b. \quad (4.4)$$

The Green's function now satisfies

$$-\frac{d}{dx} \left[p(x) \frac{dG}{dx} \right] + q(x)G = \delta(x - y). \quad (4.5)$$

The presence of the δ in the right-hand side effectively breaks the interval in two parts and therefore, while G satisfies the homogeneous equation for $x \neq y$, its form for $x < y$ is not necessarily equal to that for $x > y$. Thus, we write

$$G(x; y) = \lambda_b w_-(x)H(x - y) + \lambda_a w_+(x)H(y - x), \quad (4.6)$$

where $\lambda_{a,b}$ are constants to be suitably chosen below and w_{\pm} are two (for the time being, unspecified) solutions of the homogeneous equation⁶

$$-\frac{d}{dx} \left[p(x) \frac{dw_{\pm}}{dx} \right] + q(x)w_{\pm} = 0. \quad (4.7)$$

Upon substitution of (4.6) into (4.5), by (4.7), one is left with

$$p(y) [\lambda_a w_+(y) - \lambda_b w_-(y)] \delta'(x - y) + p(y) [\lambda_a w'_+(y) - \lambda_b w'_-(y)] \delta(x - y) = \delta(x - y), \quad (4.8)$$

where we have used the fact that $f(x)\delta(x - a) = f(a)\delta(x - a)$. Since there is no δ' in the right-hand side to balance that in the left-hand side, we must require that

$$\lambda_a w_+(y) - \lambda_b w_-(y) = 0, \quad (4.9)$$

i.e., that the Green's function be continuous at $x = y$. Balancing the other terms then leads us to

$$\lambda_a w'_+(y) - \lambda_b w'_-(y) = \frac{1}{p(y)}. \quad (4.10)$$

Equations (4.9) and (4.10) can be solved for $\lambda_{a,b}$ to find

$$\lambda_b = \frac{w_+(y)}{p(y)W(y; w_-, w_+)}, \quad \lambda_a = \frac{w_-(y)}{p(y)W(y; w_-, w_+)}, \quad (4.11)$$

in which $W(y; w_-, w_+) = w_-(y)w'_+(y) - w_+(y)w'_-(y)$ is the Wronskian of the two solutions w_{\pm} evaluated at y . To calculate this Wronskian we recall that, given $A(x)u'' + B(x)u' + C(x)u = 0$, the Wronskian of any two solutions satisfies

$$\frac{dW}{dx} = -\frac{B}{A}W, \quad (4.12)$$

so that $W = W_0 \exp(-\int(B/A)dx)$, in which W_0 is a constant depending on the two particular solutions for which the Wronskian is being calculated. In the present case $A = -p$, $B = -p'$, and therefore

$$W(x; w_-, w_+) = \frac{W_0}{p(x)}, \quad (4.13)$$

⁶Obviously, since w_{\pm} are so far unspecified, we could absorb the constants $\lambda_{a,b}$ in w_{\pm} but, if we did not show them explicitly here, we should include two undetermined multipliers in the right-hand sides of (4.19) and (4.22). The argument is cleaner by inserting them here.

so that (4.11) reduce to

$$\lambda_b = \frac{w_+(y)}{W_0}, \quad \lambda_a = \frac{w_-(y)}{W_0}, \quad (4.14)$$

and

$$G(x; y) = \frac{1}{W_0} [w_+(y)w_-(x)H(x-y) + w_-(y)w_+(x)H(y-x)]. \quad (4.15)$$

This result may be written in a more compact form by noting that, in both terms, the function w_- has as its argument the greater one of x and y , while the function w_+ has the smaller one. If we therefore let

$$x_> = \max(x, y), \quad x_< = \min(x, y), \quad (4.16)$$

we may write

$$G(x; y) = \frac{w_-(x_>)w_+(x_<)}{W_0}, \quad (4.17)$$

Let us now consider the part of the conjunct evaluated at $x = a$. Since $a < y$, $G(a; y) = \lambda_a w_+(a)$. Hence

$$[p(\partial_x G u - G u')]_{x=a} = p(a)\lambda_a [w'_+(a)u(a) - w_+(a)u'(a)]. \quad (4.18)$$

Upon comparing with the first boundary condition (4.2), we see that it would be advantageous to choose⁷

$$w_+(a) = -a_{12}, \quad w'_+(a) = a_{11}, \quad (4.19)$$

because then

$$p(a)\lambda_a [w'_+(a)u(a) - w_+(a)u'(a)] = \lambda_a p(a) [a_{11}u(a) + a_{12}u'(a)] = \lambda_a p(a)\gamma_1 = \frac{p(a)}{W_0} w_-(y)\gamma_1. \quad (4.20)$$

Equations (4.19) are two ‘initial’ (i.e., at $x = a$) conditions which fully determine the solution w_+ of (4.7). Proceeding similarly at $x = b$, we have

$$[p(\partial_x G u - G u')]_{x=b} = p(b)\lambda_b [w'_-(b)u(b) - w_-(b)u'(b)], \quad (4.21)$$

and, upon choosing

$$w_-(b) = -a_{22}, \quad w'_-(b) = a_{21}, \quad (4.22)$$

we find

$$p(b)\lambda_b [w'_-(b)u(b) - w_-(b)u'(b)] = \lambda_b p(b) [a_{21}u(b) + a_{22}u'(b)] = \lambda_b p(b)\gamma_2 = \frac{p(b)}{W_0} w_+(y)\gamma_2. \quad (4.23)$$

In conclusion, we have found that⁸

$$u(y) = \int_a^b G(x; y) f(x) dx + \gamma_1 \frac{p(a)}{W_0} w_-(y) - \gamma_2 \frac{p(b)}{W_0} w_+(y). \quad (4.24)$$

⁷A solution of the differential equation (4.7) subject to these conditions at the same end-point of the interval is guaranteed to exist when $p(a) \neq 0$ as assumed here.

⁸For the constant W_0 we have

$$W_0 = p(a) [w_-(a)w'_+(a) - w'_-(a)w_+(a)] = p(b) [w_-(b)w'_+(b) - w'_-(b)w_+(b)]$$

or, from (4.19) and (4.22),

$$W_0 = p(a) [a_{11}w_-(a) + a_{12}w'_-(a)] = -p(b) [a_{22}w'_+(b) + a_{21}w_+(b)].$$

It is useful to verify that this is indeed the correct solution. We consider the three terms separately. The first term should be a particular solution of the non-homogeneous equation. Written out in detail it is

$$u_{part}(y) = \int_a^b G(x; y) f(x) dx = w_-(y) \int_a^y \frac{w_+(x)}{W_0} f(x) dx + w_+(y) \int_y^b \frac{w_-(x)}{W_0} f(x) dx, \quad (4.25)$$

and

$$u'_{part}(y) = w'_-(y) \int_a^y \frac{w_+(x)}{W_0} f(x) dx + w'_+(y) \int_y^b \frac{w_-(x)}{W_0} f(x) dx. \quad (4.26)$$

With these relations the differential equations is readily seen to be satisfied. As for the boundary condition at a , we find

$$a_{11}u_{part}(a) + a_{12}u'_{part}(a) = [a_{11}w_+(a) + a_{12}w'_+(a)] \int_a^b \frac{w_-(x)}{W_0} f(x) dx = 0, \quad (4.27)$$

in view of (4.19). The same result holds at $y = b$. Hence, as constructed, u_{part} is the particular solution that satisfies homogeneous boundary conditions. Let us now turn to the second term of (4.24). At $y = a$, by using (4.19), we have

$$\frac{p(a)}{W_0} [a_{11}w_-(a) + a_{12}w'_-(a)] = \frac{p(a)}{W_0} [w_-(a)w'_+(a) - w'_-(a)w_+(a)] = \frac{p(a)}{W_0} W(a; w_-, w_+) = 1, \quad (4.28)$$

while, at $y = b$,

$$\frac{p(a)}{W_0} [a_{21}w_-(b) + a_{22}w'_-(b)] = 0, \quad (4.29)$$

by (4.22). Conversely, the term multiplying γ_2 in (4.24) is seen to vanish at $y = a$ and to equal 1 at $y = b$.

It can be easily verified that the preceding results can be expressed in an alternative way by defining two solutions v_1 and v_2 of the homogeneous equation (4.7) subject to the boundary conditions

$$a_{11}v_1(a) + a_{12}v'_1(a) = 1, \quad a_{21}v_1(b) + a_{22}v'_1(b) = 0, \quad (4.30)$$

and

$$a_{11}v_2(a) + a_{12}v'_2(a) = 0, \quad a_{21}v_2(b) + a_{22}v'_2(b) = 1. \quad (4.31)$$

In terms of these new functions we may write

$$u(y) = \int_a^b G(x; y) f(x) dx + \gamma_1 v_1(y) + \gamma_2 v_2(y). \quad (4.32)$$

where, from (4.24),

$$v_1(x) = \frac{p(a)}{W_0} w_-(x), \quad v_2(x) = -\frac{p(b)}{W_0} w_+(x). \quad (4.33)$$

and also

$$G(x; y) = \frac{v_1(x_>) v_2(x_<)}{\tilde{W}_0}. \quad (4.34)$$

where⁹

$$\tilde{W}_0 = p(x) W(x; v_1, v_2). \quad (4.35)$$

⁹Indeed, from the definition of $v_{1,2}$, we find

$$W(x; v_1, v_2) = -\frac{p(a)p(b)}{W_0^2} W(x; w_-, w_+).$$

Recalling that pW is a constant and evaluating e.g. at $x = a$ we find $\tilde{W}_0 = -p(a)p(b)/W_0$ while, from (4.17),

$$G(x; y) = -\frac{W_0}{p(a)p(b)} v_1(x_>) v_2(x_<).$$

Summarizing, in order to find the Green's function for (4.1), we find the two solutions of (4.7) satisfying (4.30) and (4.31), calculate the constant \tilde{W}_0 in their Wronskian from (4.35), combine them according to (4.34), after which the solution of the problem is given by (4.32).

It is easily verified from the previous developments that the Green's function that we have constructed satisfies the following properties:

- (A) From (4.17) or (4.34), G is symmetric: $G(x; y) = G(y; x)$. Thus, the variable y in (4.24) or (4.32) can be changed to x without other modifications;
- (B) From (4.9), G is continuous at $x = y$;
- (C) From (4.10), $\partial G/\partial x$ satisfies the jump condition

$$\left. \frac{\partial G}{\partial x} \right|_{y+} - \left. \frac{\partial G}{\partial x} \right|_{y-} = -\frac{1}{p(y)} \quad (4.36)$$

and, also, due to the symmetry in x and y ,

$$\left. \frac{\partial G}{\partial y} \right|_{x+} - \left. \frac{\partial G}{\partial y} \right|_{x-} = -\frac{1}{p(x)} \quad (4.37)$$

- (D) G satisfies the differential equation (4.5), written both as an equation in x and as an equation in y . Hence G is a *fundamental solution* of the equation;

It is interesting to give the following interpretation of the particular solution (4.25). We can write

$$f(x) = \sum_i f(\xi_i) [H(x - \xi_i) - H(x - \xi_{i+1})] \simeq \sum f(\xi_i) H'(x - \xi_i) \Delta\xi_i = \sum f(\xi_i) \delta(x - \xi_i) \Delta\xi_i \quad (4.38)$$

and we can write u_{part} as

$$u_{part} \simeq \sum f(\xi_i) \Delta\xi_i G(x, \xi_i), \quad (4.39)$$

which, upon comparison with (4.38), shows that $G(x, \xi_i)$ is the effect at x of a unit "load" or "source" located at ξ_i (Fig. 1).

Example. Consider the one-dimensional Helmholtz equation

$$u'' + k^2 u = f(x) \quad 0 < x < L \quad (4.40)$$

subject to

$$u(0) = 0 \quad u(L) = 0. \quad (4.41)$$

The solution of the homogeneous equation satisfying (4.30), i.e. $v_1(0) = 1, v_1(L) = 0$ is

$$v_1 = \frac{\sin k(L-x)}{\sin kL} \quad (4.42)$$

while the solution satisfying (4.31), i.e. $v_2(0) = 0, v_2(L) = 1$ is

$$v_2 = \frac{\sin kx}{\sin kL} \quad (4.43)$$

In this case $p = -1$ and therefore

$$\begin{aligned} W_0 &= pW = -(v_1 v_2' - v_1' v_2) = -\frac{k \sin k(L-x) \cos kx + k \cos k(L-x) \sin kx}{\sin^2 kL} \\ &= -\frac{k}{\sin kL}, \end{aligned} \quad (4.44)$$

which is indeed a constant. Thus

$$\begin{aligned} G &= \frac{v_1(x_>)v_2(x_<)}{W_0} = -\frac{\sin k(L-x_>)\sin kx_<}{k\sin kL} = -\frac{\cos k(L-x_>-x_<)-\cos k(L-x_>+x_<)}{2k\sin kL} \\ &= \frac{\cos k(L-|x-\xi|)-\cos k(L-x-\xi)}{2k\sin kL} \end{aligned} \quad (4.45)$$

For future reference note that the solution breaks down if $\sin kL = 0$.

4.1 An alternative derivation

The same result (4.24) can be derived in an alternative by using the method of variation of parameters.

Following the standard procedure, we let

$$u = A(x)v_1(x) + B(x)v_2(x) \quad (4.46)$$

and note that the equation may be written

$$u'' + \frac{p'}{p}u' - \frac{q}{p}u = \frac{f}{p} \quad (4.47)$$

In the usual manner we find

$$v_1A' + v_2B' = 0 \quad (4.48)$$

$$v_1'A' + v_2'B' = -\frac{f}{p} \quad (4.49)$$

from which

$$A' = \frac{v_2}{W(x;v_1,v_2)}\frac{f}{p} = \frac{v_2}{\tilde{W}_0}f, \quad B' = -\frac{v_1}{W(x;v_1,v_2)}\frac{f}{p} = -\frac{v_1}{\tilde{W}_0}f. \quad (4.50)$$

Integrating (4.50) we then find

$$A = \alpha + \frac{1}{\tilde{W}_0} \int_a^x v_2 f d\xi \quad B = \beta + \frac{1}{\tilde{W}_0} \int_x^b v_1 f d\xi \quad (4.51)$$

where the limits of integration have been chosen for later convenience and without loss of generality since the integration constants α, β are, for the moment, unspecified. Hence we have

$$u = \left[\alpha + \frac{1}{\tilde{W}_0} \int_a^x v_2(\xi) f(\xi) d\xi \right] v_1(x) + \left[\beta + \frac{1}{\tilde{W}_0} \int_x^b v_1(\xi) f(\xi) d\xi \right] v_2(x) \quad (4.52)$$

and also

$$u' = \left[\alpha + \frac{1}{\tilde{W}_0} \int_a^x v_2 f d\xi \right] v_1' + \left[\beta + \frac{1}{\tilde{W}_0} \int_x^b v_1 f d\xi \right] v_2'. \quad (4.53)$$

Now we apply the boundary conditions (4.30) and (4.31) to find

$$\alpha = \gamma_1, \quad \beta = \gamma_2. \quad (4.54)$$

so that the final form for u is

$$u = \left[\gamma_1 + \frac{1}{\tilde{W}_0} \int_a^x v_2(\xi) f(\xi) d\xi \right] v_1(x) + \left[\gamma_2 + \frac{1}{\tilde{W}_0} \int_x^b v_1(\xi) f(\xi) d\xi \right] v_2(x), \quad (4.55)$$

which may be rewritten as

$$u = \gamma_1 v_1 + \gamma_2 v_2 + \int_a^b G(x, \xi) f(\xi) d\xi \quad (4.56)$$

with $G(x, \xi)$ given by (4.34).

4.2 Remarks and applications

It is interesting to review the procedure we have used to find the Green's functions and to consider what steps require properties that may or may not be fulfilled in a specific problem.

To start with, note that in our development the quantity W_0 appears in the denominator. Since $p(x)W(w_-, w_+) = W_0$, it is clear that, if p vanishes at $x_0 \in [a, b]$, then $W_0 = 0$ unless the vanishing of p is compensated by a suitable divergence of W . Hence cases in which p vanishes at a point may have to be handled with particular care. This is one of the reasons why we have stipulated in section 4 that $p > 0$ even though there are practically important cases where p vanishes at an endpoint of the interval of interest. For instance, for the Bessel equation in an interval $0 < r < L$, we have $p(r) = r$, but, from the general theory of the Bessel functions, $W = W_0/r$ and therefore $p(r)W = W_0$ is well-defined.

4.2.1 Solvability of the non-homogeneous problem

Another important case in which the Wronskian may vanish is when the completely homogeneous problem

$$-\frac{d}{dx} \left(p \frac{dv}{dx} \right) + qv = 0 \quad (4.57)$$

$$a_{11}v(a) + a_{12}v'(a) = 0, \quad a_{21}v(b) + a_{22}v'(b) = 0, \quad (4.58)$$

has non-zero solutions. To see this, note first that, up to a multiplicative factor, only one such solution can exist. Indeed, suppose that v and \tilde{v} were two such solutions. Note that 0 is also a solution of the homogeneous problem and, since only two linearly independent solutions can exist, 0 must be expressible as a linear combination of v and \tilde{v} :

$$\alpha v + \beta \tilde{v} = 0, \quad (4.59)$$

which shows that v and \tilde{v} are proportional to each other against the hypothesis. So let us assume that there is one solution v of the completely homogeneous problem, and let w be another linearly independent solution. Then we may write

$$w_+ = \alpha v + \beta w, \quad w_- = \gamma v + \delta w. \quad (4.60)$$

Upon imposing the boundary condition (4.19) we then have

$$a_{11} [\alpha v(a) + \beta w(a)] + a_{12} [\alpha v'(a) + \beta w'(a)] = 0, \quad (4.61)$$

from which, since v satisfies the homogeneous condition,

$$a_{11} w(a) + a_{12} w'(a) = 0, \quad (4.62)$$

and similarly, at b ,

$$a_{21} w(b) + a_{22} w'(b) = 0. \quad (4.63)$$

But these relations show that w also satisfies the completely homogeneous problem and, by the previous argument, must then be proportional to v . We thus conclude that both w_+ and w_- are proportional to v and therefore to each other. In this case, as is well known, their Wronskian vanishes.

Example. We find an instance of this phenomenon in the example at the end of the previous section where, proceeding with the construction outlined before, we find

$$w_1 = \sin kx, \quad w_2 = \sin k(L-x). \quad (4.64)$$

The completely homogeneous problem has a non-zero solution if $\sin kL = 0$ and, in this case indeed

$$w_2 = \sin kL \cos kx - \cos kL \sin kx = \sin kx = -w_1 \quad (4.65)$$

and no G can be constructed, as already pointed out at the end of the last section. When this happens, the solution of the inhomogeneous problem may be impossible or undetermined depending on f . For example, suppose that $kL = \pi$. Then if, for example $f = 1$, the general solution is

$$u = \frac{L^2}{\pi^2} + A \sin kx + B \cos kx \quad (4.66)$$

and the boundary conditions $u(0) = u(L) = 0$ give

$$u(0) = \frac{L^2}{\pi^2} + B = 0, \quad u(L) = \frac{L^2}{\pi^2} - B = 0, \quad (4.67)$$

which are mutually inconsistent. Hence no solution exists for this f . On the other hand, if $f = x - L/2$ the general solution is

$$u = \frac{L^2}{\pi^2} \left(x - \frac{1}{2} L \right) + A \sin kx + B \cos kx \quad (4.68)$$

and now the boundary conditions give

$$u(0) = -\frac{L^3}{2\pi^2} - B = 0, \quad u(L) = \frac{L^3}{2\pi^2} + B = 0, \quad (4.69)$$

which are consistent, but leave A undetermined. The root of the difficulty lies in the fact that $k = \pi/L$ is an eigenvalue of the problem $u'' + k^2u = 0$, $u(0) = u(L) = 0$. The situation in cases in which this happens will be clarified by the *Fredholm alternative theorem* to be discussed later in the course.

The above arguments prove the (first part of the) following

Theorem: If the completely homogeneous boundary value problem has only the trivial solution, then the inhomogeneous problem (4.1), (4.2) with f, γ_1, γ_2 admits a solution, which is also unique.

The last part (uniqueness) can be proven as follows: let u_1, u_2 be two solutions. Then $u = u_1 - u_2$ satisfies the completely homogeneous problem. Since this has only the trivial solution, it follows that $u_1 = u_2$.

4.3 The eigenvalue Sturm-Liouville problem

In introducing the Fourier series we considered the eigenvalue problem

$$\frac{d^2w}{dx^2} = \lambda w \quad a < x < b, \quad (4.70)$$

with various two-point boundary conditions. In studying eigenfunction expansions in cylindrical coordinates we encountered Bessel's equation:

$$-\frac{d}{dr} \left(r \frac{dw}{dr} \right) + \frac{m^2}{r} w = \lambda r w \quad R_1 < r < R_2. \quad (4.71)$$

In studying the Laplacian in spherical coordinates we were led to the consider the eigenvalue problem

$$-\frac{d}{dx} \left[(1-x^2) \frac{dw}{dx} \right] + \frac{m^2}{1-x^2} w = \lambda w \quad -1 < x < 1 \quad (4.72)$$

with w regular at $x = \pm 1$. These equations were solved subject to boundary conditions of the type

$$a_{11}u(a) + a_{12}u'(a) = 0, \quad a_{21}u(b) + a_{22}u'(b) = 0, \quad (4.73)$$

where the a_{ij} are real constants.

All these examples are special cases of the general Sturm-Liouville eigenvalue problem

$$Lu = \lambda s(x)u, \quad (4.74)$$

where the operator L is defined in (4.1); we assume that s is continuous and $s > 0$. These problems are classified as regular or singular in the same way specified earlier after Eq. (4.1). Thus, the problem (4.70) is clearly regular on a finite interval, but singular if the interval is infinite. In the former case we have to supply boundary conditions at the end-points. The problem (4.71) is regular if $R_1 > 0$, but singular if $R_1 = 0$.¹⁰ In the former case one needs a specific boundary condition at R_1 , while in the latter specification of regularity at 0 is sufficient. The problem (4.72) is singular and only regularity boundary conditions at ± 1 were needed.

We have used the Sturm-Liouville eigenfunctions that we have encountered to set up orthogonal bases in the Hilbert spaces in which we were solving various problems. As we will show, there is a general theory which justifies what we have done and shows the broad generality of the method. In order to do this, we need to set up some elements of operator theory. For the time being we simply observe that, from (4.24) or (4.32), we may rewrite the problem (4.74) in the equivalent integral form

$$u(x) = \lambda \int_a^b G(x, \xi) s(\xi) u(\xi) d\xi. \quad (4.75)$$

In this way, the differential eigenvalue problem (4.74) has been transformed into an eigenvalue problem for the linear integral operator

$$Ku = \mu u \quad (4.76)$$

where $\mu = 1/\lambda$ and

$$(Ku)(x) = \int_a^b G(x, \xi) s(\xi) u(\xi) d\xi. \quad (4.77)$$

We shall see that the eigenvalue problem for K is more tractable than the original one for L .

5 Well-posed problems

According to a classification due to Hadamard, a mathematical problem is *well posed* iff

1. It has at least one solution
2. That solution is unique
3. The solution depends continuously on the data, i.e., a ‘small’ difference in the data, produces solutions that are ‘close’ to each other.

When the Green’s function exist, one can easily establish the validity of these properties. First of all, the very existence of the Green’s function permits to construct one solution, which takes care of the first point.

To prove uniqueness, let u_1, u_2 be two solutions. Then $u = u_1 - u_2$ satisfies the completely homogeneous problem and, since there are no data to drive the right-hand side of (1.7), we find $u_1 - u_2 = 0$.

To prove continuity with respect to the data, let us start by considering two inhomogeneous problems with f_1, f_2 , but the same boundary conditions. Then, from (1.7), we have

$$|u_1 - u_2| = |(G, f_1 - f_2)| \leq \|G\| \|f_1 - f_2\|, \quad (4.78)$$

where use has been made of the Schwartz inequality. Note that, here $\|G\|$ is the norm of G considered as a function of the integration variable, i.e., \mathbf{y} . Hence $\|G\|$ is still a function of \mathbf{x} but, whenever f_1 is ‘close’

¹⁰This is actually a particularly “mild” form of singularity.

to f_2 , provided $\|G\|$ is not singular, we find that u_1 will be ‘close’ to u_2 . In some cases, the argument can be strengthened by using the maximum norm in place of the L^2 norm. For example, for the Sturm-Liouville problem, we have

$$\begin{aligned} |u_1 - u_2| &= \left| \int_a^b G(x, \xi) [f_1(\xi) - f_2(\xi)] d\xi \right| \leq \int_a^b |G| |f_1 - f_2| d\xi \\ &\leq \left(\sup_{a \leq x \leq b} |f_1 - f_2| \right) \int_a^b |G| d\xi. \end{aligned} \quad (4.79)$$

If we now consider two problems with $f_1 = f_2$, but different boundary data, we have from (1.7)

$$|u_1 - u_2| = |(J[G, u_1] - (J[G, u_2])), \quad (4.80)$$

where, by suitably choosing the boundary conditions on G , the right-hand side can be made to depend only on the boundary data. For the Sturm-Liouville case this result is exceedingly simple:

$$|u_1 - u_2| = |\gamma_1^1 - \gamma_1^2| |G|, \quad (4.81)$$

and u_1 is close to u_2 if γ_1^1 is close to γ_1^2 , and similarly for the boundary condition at the other end-point. In spaces of higher dimensionality, the conjunct is an integral and an estimate similar to (4.78) can be developed.

As another example of the idea of continuous dependence on the data of the problem consider the prototypical elliptic equation, i.e. Laplace’s equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad (4.82)$$

and a similarly well-known example of a hyperbolic equation, the wave equation:

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0. \quad (4.83)$$

Here we use the same variable names for both cases to stress analogies and differences. For the same reason, in both cases, let us consider the problem in the same domain, namely

$$0 \leq x \leq 1, \quad 0 \leq y \leq L. \quad (4.84)$$

In the examples that we look at we also impose the boundary conditions

$$u(0, y) = 0, \quad u(1, y) = 0, \quad 0 \leq y \leq L. \quad (4.85)$$

The first case we consider is that of the Cauchy problem, in which the function and its normal derivative is specified on (part of) the boundary. If y were time, this would be the initial-value problem. Hence let

$$u(x, 0) = \epsilon \sin n\pi x, \quad \frac{\partial u}{\partial y}(x, 0) = 0, \quad (4.86)$$

where n is an integer, together with the boundary conditions (4.84). It is readily verified that the solution to this problem for the wave equation (4.83) is

$$u(x, y) = \epsilon \sin n\pi x \cos n\pi y. \quad (4.87)$$

This solution remains bounded no matter what n is selected. For Laplace’s equation (1), on the other hand, the solution is

$$u(x, y) = \epsilon \sin n\pi x \cosh n\pi y, \quad (4.88)$$

which blows up for large n .

Consider now a problem of the so-called boundary-value type, in which we prescribe u on the entire boundary and do not impose anything on its derivatives. Thus, in addition to (4.84), we require for example that

$$u(x, 0) = 0, \quad u(x, L) = \epsilon \sin n\pi x. \quad (4.89)$$

Now it is easy to verify that the solution of the Laplace equation (4.82) is

$$u = \epsilon \sin n\pi x \frac{\sinh n\pi y}{\sinh n\pi L}, \quad (4.90)$$

which is well-behaved no matter what n is. However, the corresponding solution for the wave equation (4.83) is

$$u = \epsilon \sin n\pi x \frac{\sin n\pi y}{\sin n\pi L}, \quad (4.91)$$

and now the solution will blow up, or nearly blow up, for all the n 's such that nL is (close to) an integer.

We conclude that the type of problems that make sense – i.e. which are *well posed* – for the wave equation are of the Cauchy type, while those that make sense for the Laplace equation are of the boundary-value type.

Until recently, it was assumed that only well-posed problems could arise in mathematical modeling. Recently, however, a number of ill-posed problems (e.g. in tomography and other inverse problems) have been shown to have major practical significance.