Derivation of Analytical Expressions for the Bit-Error Probability in Lightwave Systems with Optical Amplifiers

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***Abstract*—We describe a relatively simple derivation of the bit-error probability for a lightwave communications system using an amplitude-shift-keying (ASK) pulse modulation format and employing optical amplifiers such that amplified spontaneous emission noise dominates all other noise sources. This noise may be either polarized in the same direction as the signal or it may be unpolarized. Mathematically, it is represented as a Fourier series expansion with Fourier coefficients that are assumed to be independent Gaussian random variables. Prior to square-law detection, signal and noise passed through an optical bandpass filter. The detected current is finally filtered by temporal integration over the time slot occupied by one bit (integrate-and-dump receiver). The bit-error probability is given in a closed analytical form that is derived by the approximate evaluation of several integrals appearing in the analysis. Finally, we use our theory to derive the well-known Gaussian approximation and find that it overestimates the bit-error rate by one to two orders of magnitude. Derivations of the bit-error probability of binary ASK signals are not new, the contribution of this paper consists in its simplified approach (Fourier series expansion of the noise) and in the closed analytical form in which the final result is presented in terms of elementary functions.**

I. Introduction

We present a relatively simple derivation of the probability for the occurrence of errors for a lightwave transmission system employing optical amplifiers and using a binary amplitude shift-keying (ASK) modulation format [1], [2]. The signal is assumed to be accompanied by amplified spontaneous emission noise that is generated by the optical amplifiers and dominates all other noise sources. We model the noise as a Fourier series whose time base is chosen to coincide with the duration of a single bit. The Fourier amplitudes are independent, identically distributed Gaussian random variables. This Fourier representation has two advantages. It is well suited for modeling the influence of the optical bandpass filter through which signal and noise are passed prior to detection. Furthermore, letting the time base domain T for the Fourier series expansion coincide with the bit period has the advantage of making the sine and cosine functions of the expansion mutually orthogonal over T. Thus, when we integrate signal and noise over one bit period, after performing the squaring operation demanded by the detector, cross-product terms in the double series vanish.

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Our analysis is conceptually very simple. Complications arise only from the necessity of having to evaluate several integrals. One type of integral occurs in the process of computing probability densities by means of their characteristic functions. A second set of integrals enters the analysis when we convert probability densities to cumulative probabilities. Some of these integrals can be expressed precisely, others must be approximated. However, we obtain very accurate analytical approximations by using the method of steepest descent and the method of repeated integration by parts.

Our results agree very well with earlier published results of bit-error probabilities [2], [3]. The value of the present contribution consists in the relative simplicity of the derivation and in the fact that the desired bit-error probabilities are here expressed in closed form in terms of simple functions.

II. Outline of the Derivation of Bit-Error Probabilities

We represent the optical signal of a logical ONE (a nonreturn to zero pulse) in the time interval T of one bit by the complex representation $E_s(t) = E_1 \exp(i\omega_c t)$, ω_c is the angular carrier frequency. E_1 has a given constant value for a logical ONE and it is zero for a logical ZERO. Added to the signal pulses is amplified spontaneous emission noise $e(t)$, which we represent as a Fourier series that is also defined only for the time duration T of one bit

$$e(t) = \sum_{\nu=0}^{\infty} c_\nu e^{i\omega_\nu t} \tag{1}$$

with

$$\omega_\nu = \frac{2\pi}{T}\nu. \tag{2}$$

Note that the summation extends only over positive frequencies since we want $e(t)$ to correspond to the complex notation of the signal, the real part of these complex quantities represent the physical signal and noise. The expansion coefficients c_ν of the noise are assumed to be independent Gaussian random variables with zero mean, $\langle c_\nu \rangle = 0$, and with a variance σ that is related to the noise power in the frequency band $1/T$ that is occupied by one Fourier component.

Prior to entering the square-law detector, signal and noise are passed through an optical bandpass filter whose width is sufficient to permit the signal to pass unaltered. Its effect on the noise is to reject all frequencies outside of the passband of width B_{opt} which extends from $\nu = \nu_1$ to $\nu = \nu_1 + M$ with

$$B_{\text{opt}} = \frac{M}{T}. \tag{3}$$

Thus, the filtered noise is simply represented by

$$e(t) = \sum_{\nu=\nu_1}^{\nu_1+M} c_\nu e^{i\omega_\nu t}. \tag{4}$$

The detector produces an electrical current that is proportional to the absolute square value of the sum of signal plus noise

$$I = K\left|E_s(t) + e(t)\right|^2. \tag{5}$$

To decide whether a logical ONE or a logical ZERO has been received, the current is averaged over the time duration of one bit so that the decision is based on the quantity

$$y = \frac{1}{T}\int_0^T I\,dt. \tag{6}$$

We begin the computation of probability densities for the random variable y with the case of a logical ZERO. Setting $E(t) = 0$, we substitute (4) into (5) and (5) into (6). The squaring operation indicated in (5) converts the single sum in (4) into a double summation. However, due to the orthogonality of the functions $\exp(i\omega_\nu t)$ over the domain $0 < t < T$, only the diagonal terms in the sum remain after the integration in (6) has been performed. This leads to

$$y = K\sum_{\nu=\nu_1}^{\nu_1+M}\left(c_{r\nu}^2 + c_{i\nu}^2\right) \tag{7}$$

where the absolute square magnitude of the complex expansion coefficients has been expressed as the sum of the squares of their real and imaginary parts.

To reach our goal of deriving an expression for the probability density $W_0(y)$ for y, we introduce its characteristic function [4]

$$G_0(\zeta) = \int_0^\infty W_0(y)e^{i\zeta y}\,dy. \tag{8}$$

According to its definition, the characteristic function is defined as the average value of $\exp(i\zeta y)$. This average can be computed by introducing the probability densities for the real and imaginary parts of c appearing in (7). Since all of them are independent Gaussian random variables, the characteristic function can be expressed as a 2-m fold integral over products of 2-M Gaussian probability densities. Substituting (7) into this multiple integral and realizing that all integrals are actually identical, we obtain readily

$$G_0(\zeta) = \left\{\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp\left[-\left(\frac{1}{2\sigma^2} - iK\zeta\right)u^2\right]du\right\}^{2M} \tag{9}$$

or after performing the integration

$$G_0(\zeta) = \frac{1}{\left(1 - 2iK\sigma^2\zeta\right)^M}. \tag{10}$$

The probability density is now obtained as the inverse of the Fourier integral (8)

$$W_0(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-i\zeta y}}{\left(1 - 2iK\sigma^2\zeta\right)^M}\,d\zeta. \tag{11}$$

M integrations by parts convert this integral to the form

$$W_0(y) = \frac{1}{2\pi(M-1)!}\left(\frac{y}{2K\sigma^2}\right)^{M-1}\int_{-\infty}^{\infty}\frac{e^{-i\zeta y}}{1 - 2iK\sigma^2\zeta}\,d\zeta \tag{12}$$

which can be solved by the residue method

$$W_0(y) = \left(\frac{1}{2K\sigma^2}\right)^M \frac{y^{M-1}}{(M-1)!}e^{-(y/2K\sigma^2)}. \tag{13}$$

If the random fluctuations of the variable y exceed the decision threshold current I_d, an error is made. The probability for the occurrence of such an error is

$$P_0(I_d) = \int_{I_d}^{\infty} W_0(y)\,dy. \tag{14}$$

The integral in this expression can be expressed in terms of the incomplete gamma function [5]

$$P_0(I_d) = \frac{1}{(M-1)!}\Gamma\left(M, \frac{I_d}{2K\sigma^2}\right). \tag{15}$$

A useful and accurate approximation of the incomplete gamma function can be obtained by repeated integrations by parts of the integral in (14). Keeping the first two terms of the resulting series yields the desired approximation

$$P_0(I_d) \approx \frac{1}{(M-1)!}\left(\frac{I_d}{2K\sigma^2}\right)^{M-1} \cdot \left(1 + \frac{2K\sigma^2(M-1)}{I_d}\right)e^{-(I_d/2K\sigma^2)}. \tag{16}$$

Expression (13) for the probability density can be used to compute the average

$$\bar{I}_0 = \langle y\rangle = 2K\sigma^2 M \tag{17}$$

and the variance

$$\sigma_0^2 = M\left(2K\sigma^2\right)^2 = \frac{\bar{I}_0^2}{M} \tag{18}$$

of y.

Next, we turn to the problem of computing the probability density and the probability of error for detecting a logical ONE. We now write (5) in the form

$$I_1 = K\left[|E_1|^2 + E_s e^* + E_s^* e + |e|^2\right]. \tag{19}$$

Proceeding in close analogy to the procedure just outlined for computing $W_0(y)$, we obtain the following expression for the probability density $W_1(y)$

$$W_1(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\exp\left[-\dfrac{2K^2\sigma^2|E_1|^2\zeta^2}{1-2iK\sigma^2\zeta}\right]}{(1-2iK\sigma^2\zeta)^M} \cdot e^{i(K|E_1|^2-y)\zeta}\, d\zeta. \tag{20}$$

We list an exact solution of this integral in the appendix. Here, we present a simpler and hence more useful approximation. The terms of the integrand that are proportional to $\sigma^2\zeta$ stem from $|e|^2$ in (19) and can be interpreted as the contributions to the detector current of noise beating with noise. For reasonably large signal power the contributions of noise-noise beats are small compared to the contribution of the beats between signal and noise components, but they are not negligible and must be included in the approximations. With the help of (17) we use the approximation

$$(1-2iK\sigma^2\zeta)^M \approx e^{-i\bar{I}_0\zeta} \tag{21}$$

which holds for $MK\sigma^2 << 1$ as well as for $M \to \infty$. With this approximation we can write (20) as follows

$$W_1(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left(-\frac{2K^2\sigma^2|E_1|^2\zeta^2}{1-2iK\sigma^2\zeta}\right) e^{i(\bar{I}_1+\bar{I}_0-y)\zeta}\, d\zeta \tag{22}$$

with the signal current for logical ONES defined as

$$\bar{I}_1 = K|E_1|^2. \tag{23}$$

A good approximation to this integral can be obtained with the method of steepest descent [6] (also known as saddle point method). Without going into the details of this calculation we state its result

$$W_1(y) = \frac{1}{2}\sqrt{\frac{M}{\pi\bar{I}_0}}\left(\frac{\bar{I}_1}{(y-\bar{I}_0)^3}\right)^{1/4} \cdot \exp\left[-\frac{M}{\bar{I}_0}\left(\sqrt{y-\bar{I}_0}-\sqrt{\bar{I}_1}\right)^2\right]. \tag{24}$$

To evaluate the accuracy of this approximation we plot in Fig. 1 the logarithm of the exact solution (see Appendix) as a solid curve for $\bar{I}_1/\bar{I}_0 = 75$ and $M = 3$. The logarithm of the approximation (24) is plotted as the dotted curve. The range of applicability of the approximation (24) is somewhat restricted since the expression in (24) becomes infinite at $y = \bar{I}_0$. This pole is unphysical and hence is not included in Fig. 1, where the dotted curve is terminated at about $y = 1.1\,\bar{I}_0$. According to the physics of the problem we expect $W_0(0) = 0$.

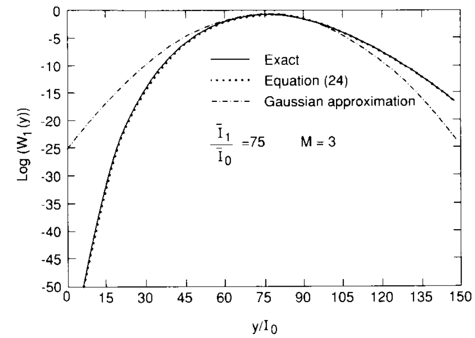


Fig. 1. Logarithm of the probability density W_1 plotted as a function of $y/\bar{I}_0$. The solid curve is computed from (20) with the help of the FFT algorithm. The jagged nearly horizontal parts of the curve are caused by loss of numerical precision. The dotted curve represents (24) while the dash-dotted curve represents the Gaussian approximation (49).

Fig. 1 shows clearly that the probability density is not symmetrical around its maximum value. If we had omitted the effect of noise-noise beats by replacing in (22) $(1 - 2iK\sigma^2\zeta) \to 1$ we would have obtained the Gaussian probability distribution that is shown as the dash-dotted line in Fig. 1. (The analytical form of this function is given in (49)). This comparison shows clearly that noise-noise beats are not negligible even when a logical ONE is being received.

The error probability for mistaking a ONE for a ZERO is given by the expression

$$P_1(I_d) = \int_0^{I_d} W_1(y)\, dy. \tag{25}$$

When we substitute (24) into (25) we obtain an integral that does not seem to have a simple solution in terms of known functions. By changing the variable of integration via the substitution

$$x = \left(\sqrt{y-\bar{I}_0} - \sqrt{\bar{I}_1}\right)^2 \tag{26}$$

we obtain

$$P_1(I_d) = \frac{1}{2}\sqrt{\frac{M}{\pi\bar{I}_0}}\,(\bar{I}_1)^{1/4}\int_{x_1}^{x_2} \frac{\exp\left(-\dfrac{M}{\bar{I}_0}x\right)}{\sqrt{x}\left(\sqrt{\bar{I}_1}-\sqrt{x}\right)}\, dx \tag{27}$$

with the integration limits

$$x_1 = \left(\sqrt{\bar{I}_1} - \sqrt{I_d - \bar{I}_0}\right)^2 \tag{28}$$

and

$$x_2 = \left(\sqrt{\bar{I}_1} - \sqrt{\bar{I}_0}\right)^2. \tag{29}$$

Even though this integral does not seem to have a simple exact solution either, a very good approximation can be obtained by realizing that the integrand contributes primarily near the lower limit $x = x_1$. By expressing the denominator of the integrand by the first two terms of its

Taylor series expansion centered at $x = x_1$, and by letting the upper limit of the resulting integral go to infinity, we obtain the following approximate solution:

$$P_1(I_d) \approx \frac{1}{2}\sqrt{\frac{\bar{I}_0}{\pi M x_1}\frac{\bar{I}_1^{1/4}}{I_{d0}^{1/4}}} \cdot \left[1 + \frac{\bar{I}_0}{4M}\frac{\sqrt{x_1} - 2\sqrt{I_{d0}}}{x_1\sqrt{I_{d0}}}\right] e^{-(Mx_1/\bar{I}_0)} \quad (30)$$

with x_1 given by (28) and I_{d0} defined as

$$I_{d0} = \bar{I}_d - \bar{I}_0. \quad (31)$$

If ZEROS and ONES are sent with equal probability one half, the total probability for an error in detecting either a ZERO or a ONE is

$$\mathrm{BER} = \tfrac{1}{2}\left[P_1(I_d) + P_0(I_d)\right]. \quad (32)$$

It is possible to express the bit-error probability in terms of the optical signal-to-noise ratio (SNR). To do this, we recall that the detected current $\bar{I}_1$ is proportional to the optical signal power of a pulse representing a logical ONE in the absence of noise. Likewise, the average current $\bar{I}_0$ is proportional (with the same proportionality constant) to the optical noise power that exists after the optical filter. The optical signal-to-noise ratio existing after the optical filter can thus be expressed as

$$s = \mathrm{SNR} = \frac{\bar{I}_1}{\bar{I}_0}. \quad (33)$$

Using (16), (17), (30), and (32) through (33), we can finally write the bit-error probability as

$$\mathrm{BER} = \frac{\exp(-pTB_{\mathrm{opt}}u)}{4\sqrt{\pi pTB_{\mathrm{opt}}u}} \cdot \left[1 + \frac{\sqrt{s} - 3\sqrt{s_d - 1}}{4pTB_{\mathrm{opt}}u\sqrt{s_d - 1}}\right]\left(\frac{s}{s_d - 1}\right)^{1/4} + \frac{(pTB_{\mathrm{opt}}s_d)^{pTB_{\mathrm{opt}}-1}}{2\Gamma(pTB_{\mathrm{opt}})}\left[1 + \frac{pTB_{\mathrm{opt}} - 1}{pTB_{\mathrm{opt}}}\frac{1}{s_d}\right] \cdot \exp(-pTB_{\mathrm{opt}}s_d). \quad (34)$$

With the abbreviations

$$s_d = \frac{I_d}{\bar{I}_0} \quad (35)$$

and

$$u = (\sqrt{s} - \sqrt{s_d - 1})^2. \quad (36)$$

In (34) we used $M = pTB_{\mathrm{opt}}$. This extension of (3) requires an explanation. So far we have assumed that the spontaneous emission noise is polarized in the same direction as the signal, so that $p = 1$. However, for unpolarized noise each of the two polarization states contributes an equal amount of statistically independent noise which doubles the number of terms in the sum (7). Thus, for unpolarized noise, M in (7) and all subsequent formulas must be replaced by 2 M, so that we have $p = 2$ in this case. Note also that the factor p is implicitly included in the definition of the average current $\bar{I}_0$. But it is important to remember that M appears only in terms originating from noise-noise beats. Beats between signal and noise (the term $E_s e^* + E_s^* e$ in (19)) do not contribute a factor M since only a single term in (4) survives the integration in (6) due to the orthogonality of the functions $\exp(i\omega_\nu t)$ over the time interval T. (We assume that $\omega_c = 2\pi\mu/T$ with integral μ.)

In addition, we expressed the factorial $(M - 1)!$ by the gamma function $\Gamma(M)$ to extend its applicability to nonintegral values.

III. Determining the Optimum Decision Level and Plots of BER

If the decision level is set optimally, the error probability should assume a minimum. Setting the derivative of (32) with respect to I_d equal to zero yields, with the help of (14) and (25), the implicit defining equation for I_d

$$W_1(I_d) = W_0(I_d). \quad (37)$$

We take the logarithm of both sides of this equation and, after rearranging terms, obtain the following implicit equation for determining $s_d = I_d/\bar{I}_0$

$$s_d = 1 + \frac{1}{4M^2 s}\left\{Ms - \frac{1}{4}\ln s - \ln A + (M - 1)\ln s_d + \frac{3}{4}\ln(s_d - 1)\right\}^2. \quad (38)$$

We have again used

$$M = pTB_{\mathrm{opt}} \quad (39)$$

and defined the parameter A appearing in (38) as

$$A = \frac{\Gamma(M)}{2M^{M-1}\sqrt{\pi M}}. \quad (40)$$

Equation (38) can be solved by direct iteration. As the signal to noise ratio s approaches infinity, (38) yields the asymptotic solution

$$\frac{s_d}{s} = \frac{1}{4}. \quad (41)$$

This is a good starting value for an iterative solution of (38) for large values of s. For the purpose of drawing curves of s_d/s as functions of s, it is advisable to start with the largest desired value of s and use (41) as the starting value for the iterative solution of (38). For successively smaller values of s, each subsequent sequence of iterations can then be started with the result of the preceding solution.

Fig. 2(a) and (b) shows curves of the relative decision level $s_d/s = I_d/\bar{I}_1$ as functions of the signal to noise ratio, $s = \mathrm{SNR}$, for $p = 1$ and $p = 2$ and for several values of

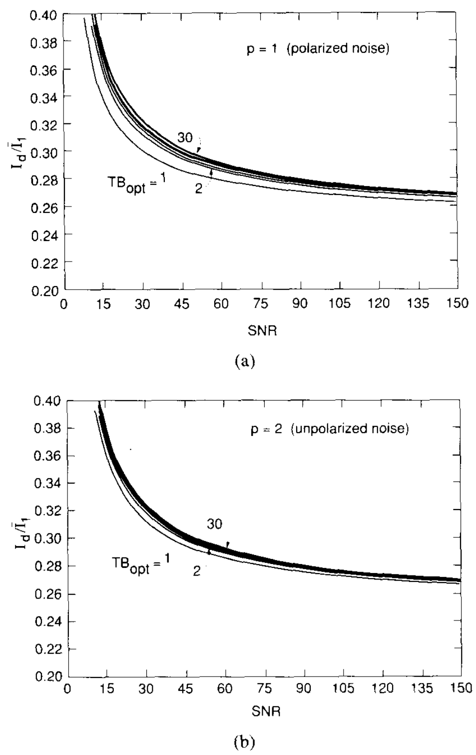


Fig. 2. Relative decision level $I_d/\bar{I}_1$ as a function of signal-to-noise ratio for several values of TB_{opt}. (a) Applies to polarized noise $p = 1$, (b) represents unpolarized noise $p = 2$.

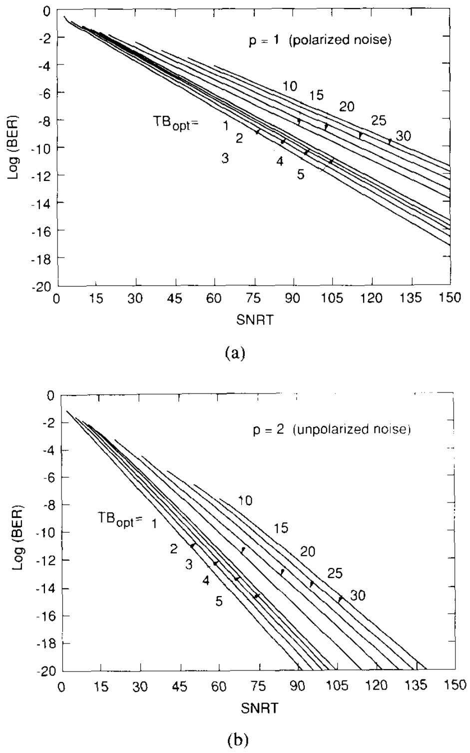


Fig. 3. The logarithm of the bit-error rate (BER) plotted as a function of the normalized signal to noise ratio SNRT. (for a definition see text) for several values of TB_{opt}. (a) Plotted for polarized noise $p = 1$, (b) applies to unpolarized noise $p = 2$.

the product TB_{opt}. The integration of the detected current over the bit-time interval T is equivalent to a low pass filter with baseband bandwidth $B_e = 1/(2T)$, so that we may interpret TB_{opt} as the ratio $B_{opt}/(2B_e)$.

Since s_d/s is nearly constant over a wide range of s values and since it does not depend critically on the values of TB_{opt}, it is possible to compute approximate values of BER from (34) without having to solve (38), by setting $s_d/s = 0.28$. However, it is not much more complicated to obtain the optimum decision level from iterative solutions of (38) and use them to compute BER from (34). This has been done to plot the curves shown in Fig. 3(a) and (b). The ordinate in these figures represents a normalized signal-to-noise ratio

$$\text{SNRT} = SNR \cdot T \cdot B_{opt}. \tag{42}$$

However, since the amount of optical noise is proportional to the bandwidth of the optical filter B_{opt} the normalized ratio can be regarded as the ratio of optical signal-to-noise power where the noise is limited to the frequency range that can pass through the electrical filter with bandwidth $2B_e = 1/T$. For this reason, we have labeled the normalized signal-to-noise ratio as SNRT which must be interpreted in the above-mentioned sense.

Fig. 3(a) and (b) pertains to $p = 1$ (polarized noise) and $p = 2$ (unpolarized noise), respectively. Each figure contains curves for a range of values of the product of the bit-time interval times the bandwidth of the optical filter TB_{opt}. To understand the apparent disparity between the two sets of figures it is important to remember that a given SNRT value for polarized noise would be reduced to one half of its value if noise is admitted to the receiver in both polarizations. Thus, a value of SNRT = 150 in Fig. 3(a) corresponds to SNRT = 75 in Fig. 3(b). Furthermore, the point SNRT = 75 and $TB_{opt} = 1$ on the curve in Fig. 3(b) correspond to the point SNRT = 150 and $TB_{opt} = 2$ in Fig. 3(a) since the amount of noise doubles when either the width of the optical filter is doubled or when the second polarization is admitted to the receiver.

The results plotted in Fig. 3(a) and (b) are in excellent agreement with corresponding results published by Henry [3].

IV. Gaussian Approximation

It is instructive to compare these results with the Gaussian approximation that is often used in the literature [7], [8].

The probability density for measuring a given current level in the presence of amplified spontaneous emission noise is given by (13) for logical ZEROS and by (24) for logical ONES. Neither of these probability densities is Gaussian. For $M = 1$ (13) is clearly an exponential function. However, by applying the central limit theorem to

(7) it is clear that $W_0(y)$ must become a Gaussian distribution as $M \to \infty$. By introducing the variable u via the substitution

$$y = \bar{I}_0 + u \tag{43}$$

and assuming that $u << 1$ it is possible to show that $W_0(u)$ does indeed approach a Gaussian distribution with the mean value (17) and the variance (18) as $M \to \infty$.

When we substitute (4) into (19) and substitute the resulting expression into (6), the central limit theorem may be invoked once more to assert that $W_1(y)$ must also approach a Gaussian distribution in the limit $M \to \infty$. This transition can also be proved by substituting

$$y = \bar{I}_1 + \bar{I}_0 + u \tag{44}$$

into (24) and letting $\bar{I}_1$ and/or M become very large. However, it is easier to invoke the central limit theorem and obtain the mean value and the variance of the distribution from the characteristic function for the probability distribution $W(y)$ which is the Fourier inverse of (20)

$$G_1(\zeta) = \frac{\exp\left(-\dfrac{2K^2\sigma^2|E_1|^2\zeta^2}{1 - 2iK\sigma^2\zeta}\right)}{(1 - 2iK\sigma^2\zeta)^M} e^{iK|E_1|^2}. \tag{45}$$

The mean is obtained as [2]

$$\langle y \rangle_1 = -i\left(\frac{\partial G_1}{\partial \zeta}\right)_{\zeta=0} = \bar{I}_1 + \bar{I}_0 \tag{46}$$

with $\bar{I}_1$ and $\bar{I}_0$ defined by (23) and (17), respectively. The variance of the distribution $W_1(y)$ is obtained as [4]

$$\sigma_1^2 = -\left(\frac{\partial^2 G_1}{\partial \zeta^2}\right)_{\zeta=0} - \langle y \rangle_1^2 = \frac{2}{M}\bar{I}_0\bar{I}_1 + \frac{\bar{I}_0^2}{M} = \frac{2}{M}\bar{I}_0\bar{I}_1 + \sigma_0^2. \tag{47}$$

Thus, we obtain for $M \to \infty$ as the Gaussian limit of (13)

$$W_{0g}(y) = \frac{1}{\sqrt{2\pi}\sigma_0} \exp\left[-\frac{(y - \bar{I}_0)^2}{2\sigma_0^2}\right] \tag{48}$$

with $\bar{I}_0$ and σ_0 defined by (17) and (18). The Gaussian limit of (24) is likewise

$$W_{1g}(y) = \frac{1}{\sqrt{2\pi}\sigma_1} \exp\left[-\frac{(y - \bar{I}_1 - \bar{I}_0)^2}{2\sigma_1^2}\right]. \tag{49}$$

We can now compute the bit-error probability in Gaussian approximation. The probability for mistaking a logical ZERO for a logical ONE is

$$P_{0g}(I_d) = \int_{I_d}^{\infty} W_{0g}(y)\, dy \approx \frac{\sigma_0}{\sqrt{2\pi}(I_d - \bar{I}_0)} \cdot \exp\left[-\frac{(I_d - \bar{I}_0)^2}{2\sigma_0^2}\right]. \tag{50}$$

The right-hand side of this expression is based on an approximation of the complementary error function that is valid for large values of its argument [3]. The probability of mistaking a logical ONE for a logical ZERO is likewise

$$P_{1g}(I_d) = \int_{-\infty}^{I_d} W_{1g}(y)\, dy \approx \frac{\sigma_1}{\sqrt{2\pi}(\bar{I}_1 + \bar{I}_0 - I_d)} \cdot \exp\left[-\frac{(\bar{I}_1 + \bar{I}_0 - I_d)^2}{2\sigma_1^2}\right]. \tag{51}$$

The optimum decision level I_d is obtained if we replace the probability densities in (37) with their corresponding Gaussian approximations. It is customary [5] to simplify the theory by equating only the exponents of the exponential functions instead of setting $W_{0g}(I_d) = W_{1g}(I_d)$. This approximation causes only a slight shift of the optimum decision level because of the very rapid variation of the exponential functions with small changes of their arguments. Equating the exponents of (48) and (49) results in

$$\frac{I_d}{\bar{I}_1} = \frac{\sigma_0}{\sigma_1 + \sigma_0} + \frac{\bar{I}_0}{\bar{I}_1} = \frac{1}{\sqrt{2\mathrm{SNR} + 1} + 1} + \frac{1}{\mathrm{SNR}}. \tag{52}$$

The exponents themselves become

$$Q = \frac{I_d - \bar{I}_0}{\sigma_0} = \frac{\bar{I}_1 + \bar{I}_0 - I_d}{\sigma_1} = \frac{\bar{I}_1}{\sigma_1 + \sigma_0}. \tag{53}$$

Introducing the signal to noise ratio $\mathrm{SNR} = \bar{I}_1/\bar{I}_0$ we may write Q as

$$Q = \frac{\mathrm{SNR}}{\sqrt{2\mathrm{SNR} + 1} + 1}\sqrt{M} \tag{54}$$

with

$$M = pTB_{\mathrm{opt}}. \tag{55}$$

If the exponents of the exponential functions in (48) and (49) are equal, it follows from (50) and (51) that $P_{0g}(I_d) = P_{1g}(I_d)$. For this reason we obtain from the Gaussian equivalent of (32) the following simple expression for the bit-error probability (bit-error rate (BER)) [7], [8]

$$\mathrm{BER} = \frac{e^{-(Q^2/2)}}{\sqrt{2\pi}Q}. \tag{56}$$

The Gaussian approximation of the bit-error rate is much simpler than the corresponding expression (34), but this gain in simplicity is bought at the expense of accuracy. Fig. 4 shows the relative decision level $I_d/\bar{I}_1$ as a function of the SNR. This curve corresponds to Fig. 2(a) and (b) of the more accurate theory. We see that the optimum decision level of the Gaussian approximation is lower than that of the more accurate theory and that $I_d/\bar{I}_1 \to 0$ as $\mathrm{SNR} \to \infty$.

The bit-error rate in the Gaussian approximation is plotted in Fig. 5(a) for polarized noise, $p = 1$ and in Fig. 5(b) for unpolarized noise $p = 2$. Comparison with the more accurate curves of Fig. 3(a) and (b) shows that the BER in the Gaussian approximation is between one and two orders of magnitude larger than the more accurate values.

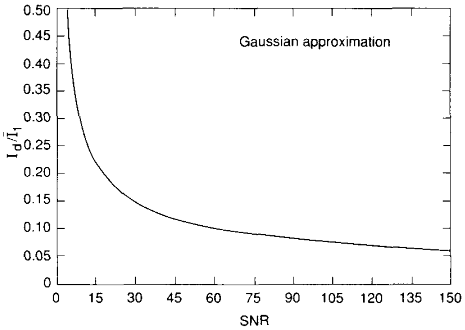
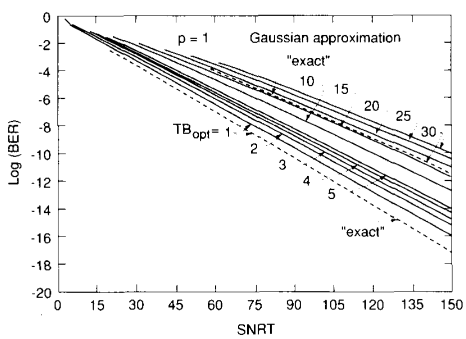
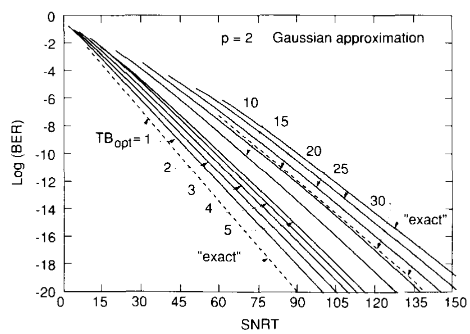


Fig. 4. Relative decision level $I_d/\bar{I}_1$ as a function of signal-to-noise ratio in Gaussian approximation.



(a)



(b)

Fig. 5. These two figures, *a* and *b*, correspond to Fig. 3(a) and (b) but were computed with the help of the Gaussian approximation.

This loss of accuracy should be kept in mind when using the Gaussian approximation for bit-error rate calculations of optical communications systems where the noise is overwhelmingly contributed by optical amplifiers.

V. Conclusions

We have studied the bit-error rate performance of an optical communications system under the assumption that the noise of the system is overwhelmingly contributed by optical amplifiers. Such systems, envisioned for very long optical fiber systems operating near the zero-dispersion wavelength would not need regenerative repeaters but would rely on analog optical amplifiers to compensate the fiber losses. However, optical amplifiers generate spontaneous emission noise that travels through the system and is amplified together with the signal. As the signal level is kept constant by repeated amplification, the amplified spontaneous emission noise grows proportionally with the number of amplifiers. Thus, for long systems the noise becomes completely dominated by this amplified spontaneous emission noise. Other noise contributions such as shot noise or avalanche noise of the detector and thermal noise in the electrical circuits is assumed to be negligible by comparison.

For comparison with established theories we use our results to derive the bit-error rate in Gaussian approximation. We find that the customary Gaussian approximation overestimates the bit-error rate of the system by one to two orders of magnitude.

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Appendix

The integral in (20) can be converted to a tabulated Laplace transform by transforming the variable of integration as follows:

$$\zeta = \frac{i}{2K\sigma^2}(z-1). \tag{A1}$$

Rearranging terms permits us to write

$$W_1(y) = \frac{\exp\left(-\frac{K|E_1|^2 + y}{2K\sigma^2}\right)}{4i\pi K\sigma^2}\int_{-i\infty}^{i\infty}\frac{1}{z^M} \cdot \exp\left(\frac{|E_1|^2}{2\sigma^2}\frac{1}{z} + \frac{y}{2K\sigma^2}z\right)dz. \tag{A2}$$

The integral in (A2) is listed in tables of Laplace transforms. With their help and with the use of (17) and (23) we obtain the following exact expression for the probability density

$$W_1(y) = \frac{M}{\bar{I}_0}\left(\frac{y}{\bar{I}_1}\right)^{(M-1)/2} \cdot \exp\left(-M\frac{y+\bar{I}_1}{\bar{I}_0}\right) I_{M-1}\left(2\frac{\sqrt{y\bar{I}_1}}{\bar{I}_0}M\right). \tag{A3}$$

In this formula, $I_{M-1}(x)$ represents the modified Bessel function of order $M-1$.

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