5

SUM OF CHI-SQUARE RANDOM VARIABLES

Define the RV $Z_2 = -Y_2$. Then the PDF of Z_2 is given by $p_{Z_2}(z) = p_{Y_2}(-z), z \le 0$. From the form of $p_Y(y)$ for central chi-square RVs, we observe that for n odd, the PDF of Z_2 is given by the PDF of Y_2 with y replaced by z and $-\sigma_2^2$ substituted for σ_2^2. For n even, the PDF of Z_2 is given by the negative of the PDF of Y_2 with y replaced by z and $-\sigma_2^2$ substituted for σ_2^2. From the form of $p_Y(y)$ for noncentral chi-square RVs, we observe that in addition to the above substitutions, $-a_2^2$ must be substituted for a_2^2. For example, for Y_2 a central chi-square RV with $2m_2$ degrees of freedom, the PDF of Z_2 is expressible as

$$\begin{aligned} p_{Z_2}(z) = p_{Y_2}(-z) &= \frac{1}{2^{m_2}\left(\sigma_2^2\right)^{m_2}\Gamma(m_2)}(-z)^{m_2-1}\exp\left(\frac{z}{2\sigma_2^2}\right) \\ &= -\frac{1}{2^{m_2}\left(-\sigma_2^2\right)^{m_2}\Gamma(m_2)}z^{m_2-1}\exp\left(-\frac{z}{2\left(-\sigma_2^2\right)}\right) \\ &= -p_{Y_2}(z)\Big|_{\sigma_2^2\to-\sigma_2^2}, \; z \le 0 \end{aligned} \tag{5.1}$$

that is, we use the expression for the PDF of Y_2 (which applies for $y \ge 0$) but substitute z for y, $-\sigma_2^2$ for σ_2^2, and then take its negative and apply it for $z \le 0$. Similarly, for Y_2 a noncentral chi-square RV with $2m_2$ degrees of freedom, the PDF of Z_2 is expressible as

$$
\begin{aligned}
p_{Z_2}(z) &= \frac{1}{2\sigma_2^2}\left(\frac{-z}{a_2^2}\right)^{(m_2-1)/2} \exp\left(-\frac{-z+a_2^2}{2\sigma_2^2}\right) I_{m_2-1}\left(\sqrt{\frac{a_2^2(-z)}{\sigma_2^4}}\right) \\
&= -\frac{1}{2(-\sigma_2^2)}\left(\frac{z}{-a_2^2}\right)^{(m_2-1)/2} \exp\left(-\frac{z-a_2^2}{2(-\sigma_2^2)}\right) I_{m_2-1}\left(\sqrt{\frac{-a_2^2 z}{(-\sigma_2^2)^2}}\right) \\
&= -p_{Y_2}(z)\Big|_{\substack{\sigma_2^2 \to -\sigma_2^2 \\ a_2^2 \to -a_2^2}}, \; z \le 0
\end{aligned}
\tag{5.2}
$$

A. Independent Central Chi-Square (+) Central Chi-Square

Define now the RV $Z = Y_1 + Y_2 = Y_1 - Z_2$. Also, for the results of Section 4A, define the notation

$$
p_Y(y) = \begin{cases} p_Y^-(y), \; y < 0 \\ p_Y^+(y), \; y \ge 0 \end{cases}
\tag{5.3}
$$

Then, it can be shown that the PDF of Z is given by

$$
p_Z(z) = \begin{cases} p_Y^-(z)\Big|_{\sigma_2^2 \to -\sigma_2^2} - p_Y^+(z)\Big|_{\sigma_2^2 \to -\sigma_2^2}, \; z \ge 0, n_2 \text{ odd} \\ p_Y^+(z)\Big|_{\sigma_2^2 \to -\sigma_2^2} - p_Y^-(z)\Big|_{\sigma_2^2 \to -\sigma_2^2}, \; z \ge 0, n_2 \text{ even} \end{cases}
\tag{5.4}
$$

Note that since Z only takes on positive (or zero) values, the PDF of Z is defined only for $z \ge 0$. Similarly, define the notation

$$
P_Y(y) = \begin{cases} P_Y^-(y), \; y < 0 \\ P_Y^+(y), \; y \ge 0 \end{cases}
\tag{5.5}
$$

Then, it can be shown from (5.4) that

$$
P_Z(z) = \begin{cases} P_Y^-(z)\Big|_{\sigma_2^2 \to -\sigma_2^2} - P_Y^+(z)\Big|_{\sigma_2^2 \to -\sigma_2^2}, \; z \ge 0, n_2 \text{ odd} \\ P_Y^+(z)\Big|_{\sigma_2^2 \to -\sigma_2^2} - P_Y^-(z)\Big|_{\sigma_2^2 \to -\sigma_2^2}, \; z \ge 0, n_2 \text{ even} \end{cases}
\tag{5.6}
$$

Before proceeding, the reader is cautioned that care must be exercised in applying (5.4) and (5.6) since in some instances the substitution $\sigma_2^2 \to -\sigma_2^2$ in the generic form of the PDF of Y might result in functions with imaginary or undefined arguments. In these instances, one is better off deriving the result for the chi-square sum directly from a convolution of the individual chi-square RV PDFs rather than from the result for the chi-square difference. In this same regard, a closed-form result for the CDF might exist for the chi-square sum RV even though it doesn't exist for the chi-square difference RV.

1. $n_1 = n_2 = 1$

$$p_Z(z) = \frac{1}{2\sigma_1\sigma_2}\exp\left(-\frac{\sigma_2^2+\sigma_1^2}{4\sigma_1^2\sigma_2^2}z\right)I_0\left(\frac{\sigma_2^2-\sigma_1^2}{4\sigma_1^2\sigma_2^2}z\right),\ z \geq 0 \tag{5.7}$$

$$P_Z(z) = 1+\exp\left(-\frac{\sigma_2^2+\sigma_1^2}{4\sigma_1^2\sigma_2^2}z\right)I_0\left(\frac{\sigma_2^2-\sigma_1^2}{4\sigma_1^2\sigma_2^2}z\right)-2Q_1\left(\frac{|\sigma_2-\sigma_1|}{2\sigma_1\sigma_2}\sqrt{z},\frac{\sigma_2+\sigma_1}{2\sigma_1\sigma_2}\sqrt{z}\right),\ z \geq 0 \tag{5.8}$$

$$\Psi_Z(\omega) = \left(\frac{1}{(1-2j\omega\sigma_1^2)(1-2j\omega\sigma_2^2)}\right)^{1/2} \tag{5.9}$$

$$E\{Z^k\} = \frac{2^{2k+1}k!(\sigma_1\sigma_2)^{2k+1}}{(\sigma_1^2+\sigma_2^2)^{k+1}}\,{}_2F_1\left(\frac{k+1}{2},\frac{k}{2}+1;1;\left(\frac{\sigma_1^2-\sigma_2^2}{\sigma_1^2+\sigma_2^2}\right)^2\right),\ k \text{ integer} \tag{5.10}$$

2. $n_1 = n_2 = 2$

$$p_Z(z) = \frac{1}{2(\sigma_2^2-\sigma_1^2)}\left[\exp\left(-\frac{z}{2\sigma_2^2}\right)-\exp\left(-\frac{z}{2\sigma_1^2}\right)\right],\ z \geq 0 \tag{5.11}$$

$$P_Z(z) = 1-\left(\frac{\sigma_2^2}{\sigma_2^2-\sigma_1^2}\right)\exp\left(-\frac{z}{2\sigma_2^2}\right)+\left(\frac{\sigma_1^2}{\sigma_2^2-\sigma_1^2}\right)\exp\left(-\frac{z}{2\sigma_1^2}\right),\ z \geq 0 \tag{5.12}$$

$$\Psi_Z(\omega) = \frac{1}{(1-2j\omega\sigma_1^2)(1-2j\omega\sigma_2^2)} \tag{5.13}$$

$$E\{Z^k\} = \frac{2^{2k+2}(k+1)!(\sigma_1\sigma_2)^{2k+2}}{(\sigma_1^2+\sigma_2^2)^{k+2}} {}_2F_1\left(\frac{k}{2}+1, \frac{k+3}{2}; \frac{3}{2}; \left(\frac{\sigma_1^2-\sigma_2^2}{\sigma_1^2+\sigma_2^2}\right)^2\right), k \text{ integer} \tag{5.14}$$

3. $n_1 = n_2 = 2m$

$$\begin{aligned} p_Z(z) &= \frac{1}{2\sigma_1^2}\exp\left(-\frac{z}{2\sigma_1^2}\right)\frac{1}{(m-1)!}\left(\frac{\sigma_1^2}{\sigma_1^2-\sigma_2^2}\right)^m \sum_{i=0}^{m-1}\frac{(2(m-1)-i)!}{i!(m-1-i)!} \\ &\times \left(\frac{\sigma_2^2}{\sigma_2^2-\sigma_1^2}\right)^{m-1-i}\left(\frac{z}{2\sigma_1^2}\right)^i \\ &+ \frac{1}{2\sigma_2^2}\exp\left(-\frac{z}{2\sigma_2^2}\right)\frac{1}{(m-1)!}\left(\frac{\sigma_2^2}{\sigma_2^2-\sigma_1^2}\right)^m \sum_{i=0}^{m-1}\frac{(2(m-1)-i)!}{i!(m-1-i)!} \\ &\times \left(\frac{\sigma_1^2}{\sigma_1^2-\sigma_2^2}\right)^{m-1-i}\left(\frac{z}{2\sigma_2^2}\right)^i, z \geq 0 \end{aligned} \tag{5.15}$$

$$\begin{aligned} P_Z(z) &= 1-\exp\left(-\frac{z}{2\sigma_1^2}\right)\frac{1}{(m-1)!}\left(\frac{\sigma_1^2}{\sigma_1^2-\sigma_2^2}\right)^m \sum_{i=0}^{m-1}\sum_{l=0}^{i}\frac{(2(m-1)-i)!}{(i-l)!(m-1-i)!} \\ &\times \left(\frac{\sigma_2^2}{\sigma_2^2-\sigma_1^2}\right)^{m-1-i}\left(\frac{z}{2\sigma_1^2}\right)^{i-l} \\ &-\exp\left(-\frac{z}{2\sigma_2^2}\right)\frac{1}{(m-1)!}\left(\frac{\sigma_2^2}{\sigma_2^2-\sigma_1^2}\right)^m \sum_{i=0}^{m-1}\sum_{l=0}^{i}\frac{(2(m-1)-i)!}{(i-l)!(m-1-i)!} \\ &\times \left(\frac{\sigma_1^2}{\sigma_1^2-\sigma_2^2}\right)^{m-1-i}\left(\frac{z}{2\sigma_2^2}\right)^{i-l}, z \geq 0 \end{aligned} \tag{5.16}$$

$$\Psi_Z(\omega) = \left(\frac{1}{(1-2j\omega\sigma_1^2)(1-2j\omega\sigma_2^2)}\right)^m \tag{5.17}$$

Note that when $\sigma_2^2 = \sigma_1^2 = \sigma^2$, then Z simply becomes a central chi-square RV with $4m$ degrees of freedom with PDF, CDF, and CF determined from (2.32), (2.33), and (2.34), respectively, with m replaced by $2m$. The moments of Z are given by

$$E\{Z^k\} = \frac{2^{2k+2m}(k+2m+1)!(\sigma_1\sigma_2)^{2k+2m}}{(2m-1)!(\sigma_1^2+\sigma_2^2)^{k+2m}} \times {}_2F_1\left(m+\frac{k}{2}, m+\frac{k+1}{2}; m+\frac{1}{2}; \left(\frac{\sigma_1^2-\sigma_2^2}{\sigma_1^2+\sigma_2^2}\right)^2\right), k \text{ integer} \tag{5.18}$$

4. $n_1 = n_2 = 2m+1$

$$p_Z(z) = \frac{\sqrt{\pi}}{2\sigma_1\sigma_2\Gamma(m+1/2)}\left(\frac{z}{2|\sigma_2^2-\sigma_1^2|}\right)^m \exp\left(-\frac{\sigma_2^2+\sigma_1^2}{4\sigma_1^2\sigma_2^2}z\right) I_m\left(\frac{\sigma_2^2-\sigma_1^2}{4\sigma_1^2\sigma_2^2}z\right), \quad z \geq 0 \tag{5.19}$$

$$P_Z(z) = ------ \tag{5.20}$$

$$\Psi_Z(\omega) = \left(\frac{1}{(1-2j\omega\sigma_1^2)(1-2j\omega\sigma_2^2)}\right)^{m+1/2} \tag{5.21}$$

$$E\{Z^k\} = \frac{2^{2(k+m)+1}(2m+k)!(\sigma_1\sigma_2)^{2(k+m)+1}}{(2m)!(\sigma_1^2+\sigma_2^2)^{2m+k+1}} \times {}_2F_1\left(m+\frac{k+1}{2}, m+1+\frac{k}{2}; m+1; \left(\frac{\sigma_1^2-\sigma_2^2}{\sigma_1^2+\sigma_2^2}\right)^2\right), k \text{ integer} \tag{5.22}$$

5. $n_1 = 2m, n_2 = 2$

Using (4.13) in (5.4), we immediately arrive at

$$p_Z(z) = \frac{1}{2\sigma_2^2}\left[\left(\frac{\sigma_2^2}{\sigma_2^2-\sigma_1^2}\right)^m \exp\left(-\frac{z}{2\sigma_2^2}\right) - \exp\left(-\frac{z}{2\sigma_1^2}\right)\right] \times \sum_{i=0}^{m-1}\frac{1}{i!}\left(\frac{\sigma_2^2}{\sigma_2^2-\sigma_1^2}\right)^{m-i}\left(\frac{z}{2\sigma_1^2}\right)^i\Bigg], z \geq 0 \tag{5.23}$$

which, of course, reduces to (5.11) when $m=1$. The corresponding expression for the CDF can be obtained by using (4.14) in (5.6) with the result

$$P_Z(z)=1+\exp\left(-\frac{z}{2\sigma_1^2}\right)\left(\frac{\sigma_1^2}{\sigma_2^2-\sigma_1^2}\right)\sum_{i=0}^{m-1}\sum_{l=0}^{i}\frac{1}{(i-l)!}\left(\frac{\sigma_2^2}{\sigma_2^2-\sigma_1^2}\right)^{m-1-i}\left(\frac{z}{2\sigma_1^2}\right)^{i-l} \\ -\left(\frac{\sigma_2^2}{\sigma_2^2-\sigma_1^2}\right)^{m}\exp\left(-\frac{z}{2\sigma_2^2}\right),\ z\geq 0 \tag{5.24}$$

$$\Psi_Z(\omega)=\frac{1}{\left(1-2j\omega\sigma_1^2\right)^m\left(1-2j\omega\sigma_2^2\right)} \tag{5.25}$$

Note that when $\sigma_2^2=\sigma_1^2=\sigma^2$, then Z simply becomes a central chi-square RV with $2(m+1)$ degrees of freedom with PDF, CDF and CF determined from (2.32), (2.33), and (2.34), respectively, with m replaced by $m+1$.

6. n_1, n_2

$$p_Z(z)=\frac{1}{2\sigma_1\sigma_2\Gamma\left(\frac{n_1+n_2}{2}\right)}\left(\frac{z}{2\sigma_1^2}\right)^{(n_1-1)/2}\left(\frac{z}{2\sigma_2^2}\right)^{(n_2-1)/2}\exp\left(-\frac{z}{2\sigma_1^2}\right) \\ \times {}_1F_1\left(\frac{n_2}{2};\frac{n_1+n_2}{2};\frac{\left(\sigma_2^2-\sigma_1^2\right)^2}{2\sigma_1^2\sigma_2^2}z\right),\ z\geq 0 \tag{5.26}$$

$$P_Z(z)=\text{------} \tag{5.27}$$

$$\Psi_Z(\omega)=\frac{1}{\left(1-2j\omega\sigma_1^2\right)^{n_1/2}\left(1-2j\omega\sigma_2^2\right)^{n_2/2}} \tag{5.28}$$

B. Dependent Central Chi-Square (+) Central Chi-Square

1. $n_1=n_2=1$

To simplify the expressions, we introduce the parameters

$$\gamma^+=\frac{\left[\left(\sigma_2^2+\sigma_1^2\right)^2-4\sigma_1^2\sigma_2^2\left(1-\rho^2\right)\right]^{1/2}}{\sigma_1^2\sigma_2^2\left(1-\rho^2\right)},\quad \beta^{\pm}=\gamma^+\pm\frac{\sigma_2^2+\sigma_1^2}{\sigma_1^2\sigma_2^2\left(1-\rho^2\right)} \tag{5.29}$$

Note that $\beta^+ \geq 0$ but $\beta^- \leq 0$. Then,

$$p_Z(z) = \frac{1}{2\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left(-\frac{1}{4}(\beta^+ - \gamma^+)z\right) I_0\left(\frac{1}{4}\gamma^+ z\right), \; z \geq 0 \tag{5.30}$$

$$P_Z(z) = 1 + \exp\left(-\frac{1}{4}(\beta^+ - \gamma^+)z\right) I_0\left(\frac{1}{4}\gamma^+ z\right) - 2Q_1\left(\frac{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\sqrt{1-\rho^2}}}{2\sigma_1\sigma_2\sqrt{1-\rho^2}}\sqrt{z}, \frac{\sqrt{\sigma_1^2 + \sigma_2^2 + 2\sigma_1\sigma_2\sqrt{1-\rho^2}}}{2\sigma_1\sigma_2\sqrt{1-\rho^2}}\sqrt{z}\right), \; z \geq 0 \tag{5.31}$$

$$\Psi_Z(\omega) = \left(\frac{1-\rho^2}{\left(1-2j\omega(1-\rho^2)\sigma_1^2\right)\left(1-2j\omega(1-\rho^2)\sigma_2^2\right) - \rho^2}\right)^{1/2} \tag{5.32}$$

$$E\{Z^k\} = \frac{2^{2k+1}k!}{\sigma_1\sigma_2\sqrt{1-\rho^2}\,(\beta^+ - \gamma^+)^{k+1}} \, {}_2F_1\left(\frac{k+1}{2}, \frac{k}{2}+1; 1; \left(\frac{\gamma^+}{\beta^+ - \gamma^+}\right)^2\right), \; k \text{ integer} \tag{5.33}$$

2. $n_1 = n_2 = 2$

$$p_Z(z) = \frac{1}{2\sigma_1^2\sigma_2^2(1-\rho^2)\gamma^+}\left[\exp\left(\frac{1}{4}\beta^- z\right) - \exp\left(-\frac{1}{4}\beta^+ z\right)\right], \; z \geq 0 \tag{5.34}$$

$$P_Z(z) = \frac{2}{\sigma_1^2\sigma_2^2(1-\rho^2)\gamma^+}\left\{-\frac{1}{\beta^-}\left[1 - \exp\left(\frac{1}{4}\beta^- z\right)\right] - \frac{1}{\beta^+}\left[1 - \exp\left(-\frac{1}{4}\beta^+ z\right)\right]\right\}, \; z \geq 0 \tag{5.35}$$

$$\Psi_Z(\omega) = \frac{1-\rho^2}{\left(1-2j\omega(1-\rho^2)\sigma_1^2\right)\left(1-2j\omega(1-\rho^2)\sigma_2^2\right) - \rho^2} \tag{5.36}$$

$$E\{Z^k\} = \frac{2^{2(k+1)}(k+1)!}{\sigma_1^2\sigma_2^2(1-\rho^2)(\beta^+ - \gamma^+)^{k+2}} \, {}_2F_1\left(\frac{k}{2}+1, \frac{k+3}{2}; \frac{3}{2}; \left(\frac{\gamma^+}{\beta^+ - \gamma^+}\right)^2\right),$$

k integer

(5.37)

3. $n_1 = n_2 = 2m$

$$p_Z(z) = \frac{z^{m-1}}{(m-1)!\left[2\sigma_1^2\sigma_2^2(1-\rho^2)\gamma^+\right]^m}\left[\exp\left(\frac{1}{4}\beta^- z\right)\sum_{i=0}^{m-1}\frac{(m+i-1)!}{i!(m-i-1)!}\left(\frac{2}{\gamma^+ z}\right)^i\right.$$
$$\left. +(-1)^m \exp\left(-\frac{1}{4}\beta^+ z\right)\sum_{i=0}^{m-1}\frac{(m+i-1)!}{i!(m-i-1)!}\left(\frac{2}{\gamma^+ z}\right)^i\right], z \geq 0 \qquad (5.38)$$

$$P_Z(z) = \frac{1}{(m-1)!\left(2\sigma_1^2\sigma_2^2(1-\rho^2)\gamma^+\right)^m}\sum_{i=0}^{m-1}\frac{(-1)^i(m+i-1)!}{i!}\left(\frac{2}{\gamma^+}\right)^i$$
$$\times\left\{\left[\frac{1}{(-\beta^-/4)^{m-i}} - \exp\left(\frac{1}{4}\beta^- z\right)\sum_{l=0}^{m-i-1}\frac{1}{(m-i-l-1)!(-\beta^-/4)^{l+1}}z^{m-i-l-1}\right]\right.$$
$$\left. +(-1)^{m-i}\left[\frac{1}{(\beta^+/4)^{m-i}} - \exp\left(-\frac{1}{4}\beta^+ z\right)\sum_{l=0}^{m-i-1}\frac{1}{(m-i-l-1)!(\beta^+/4)^{l+1}}z^{m-i-l-1}\right]\right\},$$

$z \geq 0$

(5.39)

$$\Psi_Z(\omega) = \left(\frac{1-\rho^2}{\left(1-2j\omega(1-\rho^2)\sigma_1^2\right)\left(1-2j\omega(1-\rho^2)\sigma_2^2\right)-\rho^2}\right)^m \qquad (5.40)$$

4. $n_1 = n_2 = 2m+1$

$$p_Z(z) = \frac{\sqrt{\pi/2}}{\left[2\sigma_1^2\sigma_2^2(1-\rho^2)\right]^{m+1/2}\Gamma(m+1/2)}\left(\frac{z}{(\gamma^+)^2}\right)^m \exp\left(-\frac{1}{4}(\beta^+ - \gamma^+)z\right)$$
$$\times I_m\left(\frac{1}{4}\gamma^+ z\right), z \geq 0 \qquad (5.41)$$

$$P_Z(z) = ------ \tag{5.42}$$

$$\Psi_Z(\omega) = \left(\frac{1-\rho^2}{\left(1-2j\omega(1-\rho^2)\sigma_1^2\right)\left(1-2j\omega(1-\rho^2)\sigma_2^2\right)-\rho^2} \right)^{m+1/2} \tag{5.43}$$

$$\begin{aligned} E\{Z^k\} &= \frac{2^{2(m+k)+1}(2m+k)!}{\left(\sigma_1\sigma_2\sqrt{1-\rho^2}\right)^{2m+1}\left(\beta^+-\gamma^+\right)^{2m+1+k}} \\ &\times {}_2F_1\left(m+\frac{k+1}{2}, m+1+\frac{k}{2}; m+1; \left(\frac{\gamma^+}{\beta^+-\gamma^+}\right)^2\right), k \text{ integer} \end{aligned} \tag{5.44}$$

C. Independent Noncentral Chi-Square (+) Central Chi-Square

Define $Z = Y_1 + Y_2$ where Y_1 and Y_2 are independent noncentral and central chi-square distributed RVs with n_1 and n_2 degrees of freedom, respectively.

1. $n_1 = n_2 = n$

$$\begin{aligned} p_Z(z) &= \frac{1}{2\sigma_1^2}\left(\frac{\sigma_1}{\sigma_2}\right)^n \left(\frac{z}{a_1^2}\right)^{(n-1)/2} \exp\left(-\frac{z+a_1^2}{2\sigma_1^2}\right) \\ &\times \sum_{i=0}^{\infty} \frac{\Gamma(n/2+i)}{i!\Gamma(n/2)} \left(\frac{\sqrt{z}(\sigma_2^2-\sigma_1^2)}{a_1\sigma_2^2}\right)^i I_{n+i-1}\left(\frac{\sqrt{z}a_1}{\sigma_1^2}\right), z \geq 0 \end{aligned} \tag{5.45}$$

$$P_Z(z) = \left(\frac{\sigma_1}{\sigma_2}\right)^n \sum_{i=0}^{\infty} \frac{\Gamma(n/2+i)}{i!\Gamma(n/2)} \left(\frac{\sigma_2^2-\sigma_1^2}{\sigma_2^2}\right)^i \left[1 - Q_{n+i}\left(\frac{a_1}{\sigma_1}, \frac{\sqrt{z}}{\sigma_1}\right)\right], z \geq 0 \tag{5.46}$$

$$\Psi_Z(\omega) = \left(\frac{1}{(1-2j\omega\sigma_1^2)(1-2j\omega\sigma_2^2)}\right)^{n/2} \exp\left(\frac{j\omega a_1^2}{1-2j\omega\sigma_1^2}\right) \tag{5.47}$$

2. $n_1 = 2m_1, n_2 = 2m_2$

$$p_Z(z) = \frac{1}{2\sigma_1^2}\left(\frac{\sigma_1}{\sigma_2}\right)^{2m_2}\left(\frac{z}{a_1^2}\right)^{(m_1+m_2-1)/2}\exp\left(-\frac{z+a_1^2}{2\sigma_1^2}\right)$$
$$\times\sum_{i=0}^{\infty}\frac{\Gamma(m_2+i)}{i!\,\Gamma(m_2)}\left(\frac{\sqrt{z}(\sigma_2^2-\sigma_1^2)}{a_1\sigma_2^2}\right)^i I_{m_1+m_2+i-1}\left(\frac{a_1\sqrt{z}}{\sigma_1^2}\right),\ z\ge 0 \tag{5.48}$$

$$P_Z(z) = \left(\frac{\sigma_1}{\sigma_2}\right)^{2m_2}\sum_{i=0}^{\infty}\frac{\Gamma(m_2+i)}{i!\,\Gamma(m_2)}\left(\frac{\sigma_2^2-\sigma_1^2}{\sigma_2^2}\right)^i\left[1-Q_{m_1+m_2+i}\left(\frac{a_1}{\sigma_1},\frac{\sqrt{z}}{\sigma_1}\right)\right],\ z\ge 0 \tag{5.49}$$

$$\Psi_Z(\omega) = \frac{1}{(1-2j\omega\sigma_1^2)^{m_1}(1-2j\omega\sigma_2^2)^{m_2}}\exp\left(\frac{j\omega a_1^2}{1-2j\omega\sigma_1^2}\right) \tag{5.50}$$

3. $n_1 = 2m, n_2 = 2$

For $\sigma_2^2 > \sigma_1^2$, applying (5.4) and (5.6) to (4.35) and (4.36) gives

$$p_Z(z) = \frac{1}{2\sigma_2^2}\left(\frac{\sigma_2^2}{\sigma_2^2-\sigma_1^2}\right)^m\exp\left(-\frac{z}{2\sigma_2^2}\right)\exp\left(\frac{a_1^2}{2(\sigma_2^2-\sigma_1^2)}\right)$$
$$\times\left[1-Q_m\left(\frac{a_1}{\sigma_1}\sqrt{\frac{\sigma_2^2}{\sigma_2^2-\sigma_1^2}},\sqrt{\frac{z(\sigma_2^2-\sigma_1^2)}{\sigma_1^2\sigma_2^2}}\right)\right],\ z\ge 0 \tag{5.51}$$

$$P_Z(z) = 1-Q_m\left(\frac{a_1}{\sigma_1},\frac{\sqrt{z}}{\sigma_1}\right)-\left(\frac{\sigma_2^2}{\sigma_2^2-\sigma_1^2}\right)^m\exp\left(-\frac{z}{2\sigma_2^2}\right)\exp\left(\frac{a_1^2}{2(\sigma_2^2-\sigma_1^2)}\right)$$
$$\times\left[1-Q_m\left(\frac{a_1}{\sigma_1}\sqrt{\frac{\sigma_2^2}{\sigma_2^2-\sigma_1^2}},\sqrt{\frac{z(\sigma_2^2-\sigma_1^2)}{\sigma_1^2\sigma_2^2}}\right)\right],\ z\ge 0 \tag{5.52}$$

$$\Psi_Z(\omega) = \frac{1}{(1-2j\omega\sigma_1^2)^m(1-2j\omega\sigma_2^2)}\exp\left(\frac{j\omega a_1^2}{1-2j\omega\sigma_1^2}\right) \tag{5.53}$$

For the limited case of $\sigma_2^2 = \sigma_1^2 = \sigma^2$, one can use the series expansion of the generalized Marcum Q-function, namely,

$$1-Q_k(\alpha,\beta)=\exp\left(-\frac{\alpha^2+\beta^2}{2}\right)\sum_{l=k}^{\infty}\left(\frac{\beta}{\alpha}\right)^l I_l(\alpha\beta) \qquad (5.54)$$

in (5.51) to arrive at the results

$$p_Z(z)=\frac{1}{2\sigma^2}\left(\frac{\sqrt{z}}{a_1}\right)^m \exp\left(-\frac{z+a_1^2}{2\sigma^2}\right)I_m\left(\frac{a_1\sqrt{z}}{\sigma^2}\right),\ z\geq 0 \qquad (5.55)$$

$$P_Z(z)=1-Q_{m+1}\left(\frac{a_1}{\sigma},\frac{\sqrt{z}}{\sigma}\right),\ z\geq 0 \qquad (5.56)$$

These results could also have been immediately obtained by noting that, for this limiting case, Z is simply a noncentral chi-square RV with $2(m+1)$ degrees of freedom and the value of a_1 is still obtained from (1.14) since the addition of the central chi-square RV to Y_1 does not change this value.

For $\sigma_2^2<\sigma_1^2$, the form of the PDF and CDF change with respect to those given in (5.51) and (5.52). Still we can apply the series expansion of the generalized Marcum Q-function to these equations keeping in mind that now the arguments of the Marcum Q-function, α and β, will be purely imaginary. Carrying out the algebra and recalling that $I_l(-x)=(-1)^l I_l(x)$, we obtain

$$p_Z(z)=\frac{1}{2\sigma_2^2}\exp\left(-\frac{z+a_1^2}{2\sigma_1^2}\right)\sum_{l=m}^{\infty}\left(\frac{\sigma_2^2-\sigma_1^2}{\sigma_2^2}\right)^{l-m}\left(\frac{\sqrt{z}}{a_1}\right)^l I_l\left(\frac{a_1\sqrt{z}}{\sigma_1^2}\right),\ z\geq 0 \qquad (5.57)$$

and

$$P_Z(z)=1-Q_m\left(\frac{a_1}{\sigma_1},\frac{\sqrt{z}}{\sigma_1}\right)-\exp\left(-\frac{z+a_1^2}{2\sigma_1^2}\right)\sum_{l=m}^{\infty}\left(\frac{\sigma_2^2-\sigma_1^2}{\sigma_2^2}\right)^{l-m}\left(\frac{\sqrt{z}}{a_1}\right)^l I_l\left(\frac{a_1\sqrt{z}}{\sigma_1^2}\right),\ z\geq 0 \qquad (5.58)$$

which can also be put into the form

$$P_Z(z)=\left(\frac{\sigma_1}{\sigma_2}\right)^2\sum_{i=0}^{\infty}\left(\frac{\sigma_2^2-\sigma_1^2}{\sigma_2^2}\right)^i\left[1-Q_{m_1+1+i}\left(\frac{a_1}{\sigma_1},\frac{\sqrt{z}}{\sigma_1}\right)\right],\ z\geq 0 \qquad (5.59)$$

D. Independent Noncentral Chi-Square (+) Noncentral Chi-Square

Define $Z = Y_1 + Y_2$ where Y_1 and Y_2 are independent noncentral chi-square distributed RVs with n_1 and n_2 degrees of freedom, respectively.

1. $n_1 = n_2 = n$

$$p_Z(z) = \frac{1}{2\sigma_1^2}\left(\frac{\sigma_1}{\sigma_2}\right)^n \left(\frac{z}{a_1^2}\right)^{(n-1)/2} \exp\left(-\frac{z}{2\sigma_1^2}\right)\exp\left[-\frac{1}{2}\left(\frac{a_1^2}{\sigma_1^2}+\frac{a_2^2}{\sigma_2^2}\right)\right]$$
$$\times \sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\frac{\Gamma(n/2+i+l)}{i!l!\Gamma(n/2+l)}\left(\frac{\sqrt{z}a_2^2\sigma_1^2}{2a_1\sigma_2^4}\right)^l \left(\frac{\sqrt{z}(\sigma_2^2-\sigma_1^2)}{a_1\sigma_2^2}\right)^i I_{n+i+l-1}\left(\frac{\sqrt{z}a_1}{\sigma_1^2}\right),\ z \ge 0 \tag{5.60}$$

$$P_Z(z) = \left(\frac{\sigma_1}{\sigma_2}\right)^n \exp\left(-\frac{a_2^2}{2\sigma_2^2}\right)\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\frac{\Gamma(n/2+i+l)}{i!l!\Gamma(n/2+l)}\left(\frac{a_2^2\sigma_1^2}{2\sigma_2^2}\right)^l\left(\frac{\sigma_2^2-\sigma_1^2}{\sigma_2^2}\right)^i$$
$$\times\left[1-Q_{n+i+l}\left(\frac{a_1}{\sigma_1},\frac{\sqrt{z}}{\sigma_1}\right)\right],\ z\ge 0 \tag{5.61}$$

$$\Psi_Z(\omega) = \left(\frac{1}{(1-2j\omega\sigma_1^2)(1-2j\omega\sigma_2^2)}\right)^{n/2}\exp\left(\frac{j\omega a_1^2}{1-2j\omega\sigma_1^2}\right)\exp\left(\frac{j\omega a_2^2}{1-2j\omega\sigma_2^2}\right) \tag{5.62}$$

2. $n_1 = 2m_1, n_2 = 2m_2$

$$p_Z(z) = \frac{1}{2\sigma_1^2}\left(\frac{\sigma_1}{\sigma_2}\right)^{2m_2}\left(\frac{z}{a_1^2}\right)^{(m_1+m_2-1)/2}\exp\left(-\frac{z}{2\sigma_1^2}\right)\exp\left[-\frac{1}{2}\left(\frac{a_1^2}{\sigma_1^2}+\frac{a_2^2}{\sigma_2^2}\right)\right]$$
$$\times\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\frac{\Gamma(m_2+i+l)}{i!l!\Gamma(m_2+l)}\left(\frac{\sqrt{z}a_2^2\sigma_1^2}{2a_1\sigma_2^4}\right)^l\left(\frac{\sqrt{z}(\sigma_2^2-\sigma_1^2)}{a_1\sigma_2^2}\right)^i I_{m_1+m_2+i+l-1}\left(\frac{\sqrt{z}a_1}{\sigma_1^2}\right), \tag{5.63}$$
$$z \ge 0$$

$$P_Z(z) = \left(\frac{\sigma_1}{\sigma_2}\right)^{2m_2} \exp\left(-\frac{a_2^2}{2\sigma_2^2}\right) \sum_{i=0}^{\infty} \sum_{l=0}^{\infty} \frac{\Gamma(m_2+i+l)}{i!l!\Gamma(m_2+l)} \left(\frac{a_2^2\sigma_1^2}{2\sigma_2^2}\right)^l \left(\frac{\sigma_2^2-\sigma_1^2}{\sigma_2^2}\right)^i \times\left[1 - Q_{m_1+m_2+i+l}\left(\frac{a_1}{\sigma_1}, \frac{\sqrt{z}}{\sigma_1}\right)\right], z \geq 0 \quad (5.64)$$

$$\Psi_Z(\omega) = \frac{1}{\left(1-2j\omega\sigma_1^2\right)^{m_1}\left(1-2j\omega\sigma_2^2\right)^{m_2}} \exp\left(\frac{j\omega a_1^2}{1-2j\omega\sigma_1^2}\right) \exp\left(\frac{j\omega a_2^2}{1-2j\omega\sigma_2^2}\right) \quad (5.65)$$