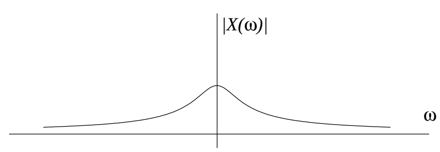
Recitation problem Set 10

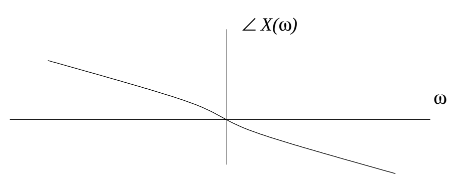
1. Sketch the magnitude and phase spectra for

$$x(t) = e^{-(t-1)}u(t-1)$$

$$\begin{aligned}
X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{1}^{\infty} e^{-(t-1)}e^{-j\omega t}dt \\
&= \int_{1}^{\infty} e^{-(1+j\omega)t}dt = \frac{-e}{1+j\omega}e^{-(1+j\omega)t}|_1^{\infty} \\
&= \frac{e}{1+j\omega}e^{-(1+j\omega)} = \frac{e^{-j\omega}}{1+j\omega} \\
|X(\omega)| &= \frac{1}{\sqrt{1+\omega^2}} \\
\angle X(\omega) &= \angle e^{-j\omega} - \angle 1 + j\omega = \underbrace{-\omega}_{0\to-\infty} - \underbrace{\arctan\omega}_{0\to-\pi/2}
\end{aligned}$$

Sketch is kind of messy.





2. Sketch the spectra for

$$x(t) = 3 + e^{-(t-1)}u(t-1)$$

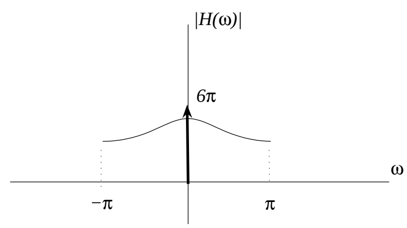
From 1.

$$\begin{aligned}
X(\omega) &= f[3] + \frac{e^{-j\omega}}{1+j\omega} \\
f[3] &= \int_{-\infty}^{\infty} 3e^{-j\omega t}dt = ...trouble
\end{aligned}$$

But if we guess $\hat{X}(\omega) = 6\pi\delta(\omega)$ then

$$\begin{aligned}\hat{x}(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} 6\pi\delta(\omega)e^{j\omega t}d\omega = 3 \\ \Rightarrow \quad X(\omega) &= 6\pi\delta(\omega) + \frac{e^{-j\omega}}{1+j\omega}\end{aligned}$$

Treat $|X(\omega)|$ as the sum of the magnitude, since the two terms are "non-overlapping", except at $\omega = 0$.



Also, since $6\pi\delta(\omega)$ is real with $\angle 6\pi = 0$, and is nonzero at only one point, we ignore its angle.

3. Derive the Fourier Transform of $x(t) = \cos(3t)$

Direct approach leads to problems:

$$X(\omega) = \int_{-\infty}^{\infty} \cos(3t)e^{-j\omega t}dt = ...?$$

Integral dos not converge in usual sense.
First compute Fourier Series for $x(t)$:

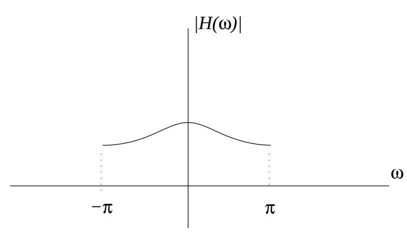
$$\begin{aligned}T_o &= \frac{2\pi}{3},\ \omega_o = 3 \\ \cos(3t) &= \frac{1}{2}e^{j3t} + \frac{1}{2}e^{-j3t} \\ \Rightarrow \quad x_1 &= x_{-1} = \frac{1}{2}, \textit{all other } x_k's = 0 \\ \Rightarrow \quad X(\omega) &= \sum_{k=-\infty}^{\infty} 2\pi x_k \delta(\omega - k\omega_o) \\ &= \pi\delta(\omega-3) + \pi\delta(\omega+3)\end{aligned}$$

4. What are the filtering characteristics of the system

$$y[n] - \frac{1}{2}y[n-1] = x[n]$$

Recall

$$\begin{aligned}
h[n] &= (\frac{1}{2})^n u[n] \\
\Rightarrow \quad H(\omega) &= \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} = \sum_{n=0}^{\infty} (\frac{1}{2})^n e^{-j\omega n} \\
&= \sum_{n=0}^{\infty} (\frac{1}{2}e^{-j\omega})^n = \frac{1}{1-\frac{1}{2}e^{-j\omega}} \\
|H(\omega)| &= \frac{1}{|1-\frac{1}{2}e^{-j\omega}|} = \frac{1}{\sqrt{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{2}e^{j\omega})}} \\
&= \frac{1}{\sqrt{1-\frac{1}{2}\cos(\omega)+\frac{1}{4}}} \\
|H(0)| &= \frac{1}{\sqrt{3/4}}, \quad |H(\pi)| = \frac{1}{\sqrt{7/4}}
\end{aligned}$$



"Low-Pass"