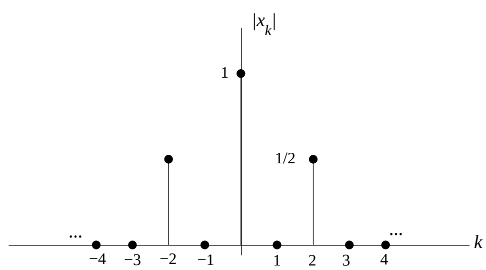
Recitation problem Set 8

1. An LTI system with frequency response function

$$H(\omega) = \frac{-1 + j\omega}{1 + j\omega}$$

has a periodic input signal $x(t)$. The fundamental period of $x(t)$ is $T_o = 2$ and the magnitude spectrum of $x(t)$ is shown below.



Sketch the magnitude spectrum of the output $y(t)$.

For $x(t)$, $T_o = 2 \quad \Rightarrow \quad \omega_o = \pi$, and the Fourier series for $x(t)$ is

$$x(t) = x_o + x_2\, e^{j2\pi t} + x_{-2}\, e^{-j2\pi t}$$

Where $\mid x_o \mid= 1$, $\mid x_2 \mid=\mid x_{-2} \mid= \frac{1}{2}$.
By linearity and the eigenfunction property,

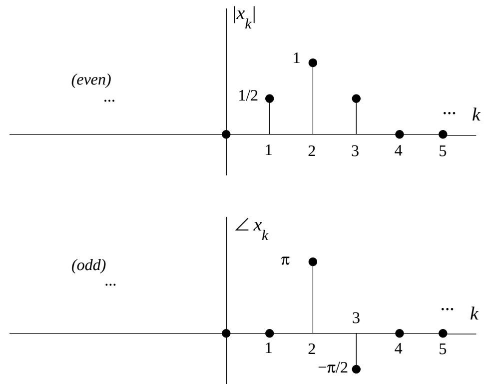
$$y(t) = H(0)x_o + H(2\pi)x_2\, e^{j2\pi t} + H(-2\pi)x_{-2}\, e^{-j2\pi t}$$

and

$$\begin{aligned}
\mid H(0)x_o \mid &=\mid H(0) \mid \cdot \mid x_o \mid= 1 \\
\mid H(2\pi)x_2 \mid &=\mid H(2\pi) \mid \cdot \mid x_2 \mid= 1 \cdot \frac{1}{2} = \frac{1}{2} \\
\mid H(-2\pi)x_{-2} \mid &=\mid H^*(2\pi)\, x_2^* \mid= 1 \cdot \frac{1}{2} = \frac{1}{2}
\end{aligned}$$

Thus, the magnitude spectrum of $y(t)$ is identical to that for $x(t)$.

2. A periodic signal $x(t)$ with $T_o = 5$ has the magnitude and phase spectra shown below



What is x(t)?

$$
\begin{array}{l}
| x_o |= 1,\ \angle x_o = 0 \quad \Rightarrow \quad x_o = 0 \\
| x_1 |= \frac{1}{2},\ \angle x_1 = 0 \quad \Rightarrow \quad x_1 = \frac{1}{2} \quad \Rightarrow x_{-1} = x_1^* = \frac{1}{2} \\
| x_2 |= 1,\ \angle x_2 = \pi \quad \Rightarrow \quad x_2 = -1 \quad \Rightarrow x_{-2} = x_2^* = -1 \\
| x_3 |= \frac{1}{2},\ \angle x_3 = \frac{-\pi}{2} \quad \Rightarrow \quad x_3 = \frac{-j}{2} = \frac{1}{2j} \quad \Rightarrow x_{-3} = x_3^* = \frac{-1}{2j}
\end{array}
$$

Thus, since $\omega_o = \frac{2\pi}{T_o} = \frac{2\pi}{5}$,

$$
\begin{aligned}
x(t) &= \frac{1}{2}e^{j\frac{2\pi}{5}t} - e^{j\frac{4\pi}{5}t} + \frac{1}{2j}e^{j\frac{6\pi}{5}t} + \frac{1}{2}e^{-j\frac{2\pi}{5}t} - e^{-j\frac{4\pi}{5}t} - \frac{1}{2j}e^{-j\frac{6\pi}{5}t} \\
&= \cos(\frac{2\pi}{5}t) - 2\cos(\frac{4\pi}{5}t) + \sin(\frac{6\pi}{5}t)
\end{aligned}
$$

3. Suppose that the continuous-time signal $x(t)$ has the Fourier series representation

$$x(t) = \sum_{k=-\infty}^{\infty} (\frac{1}{3})^{|k|} e^{jk4\pi t}$$

Compute the Fourier series representation for $z(t) = x(2t-1)$.

For $x(t)$, $\omega_o = 4\pi \quad \Rightarrow \quad T_o = \frac{1}{2}$ and $x_k = (\frac{1}{3})^{|k|}$
For $z(t)$, the shift does not change the period and the scale-by-2 gives $\hat{T}_o = \frac{T_o}{2}$, $\hat{\omega}_o = 2\omega_o$.
The F.S. coefficients for $z(t)$ are

$$
\begin{aligned}
\hat{x}_k &= \frac{1}{\hat{T}_o}\int_0^{\hat{T}_o} z(t)e^{-jk\hat{\omega}_o t}\, dt \\
&= \frac{1}{\frac{1}{2}T_o}\int_0^{\frac{1}{2}T_o} x(2t-1)e^{-jk2\omega_o t}dt
\end{aligned}
$$

Let $\tau = 2t - 1$

$$\begin{aligned}
\hat{x}_k &= \frac{2}{T_o}\int_{-1}^{T_o-1} x(\tau) e^{-jk2\omega_o(\frac{\tau+1}{2})} \ \frac{1}{2}\ d\tau \\
&= \underbrace{e^{-jk\omega_o}}_{=e^{-jk4\pi}=1\forall k} \underbrace{\frac{1}{T_o}\int_{-1}^{T_o-1} x(\tau) e^{-jk\omega_o\tau}\ d\tau}_{=x_k=(\frac{1}{3})^{|k|}} \\
\Rightarrow \quad &= (\frac{1}{3})^{|k|}
\end{aligned}$$

Thus,

$$z(t) = \sum_{k=-\infty}^{\infty} (\frac{1}{3})^{|k|} e^{jk8\pi t}$$