1.1Fourier transform and Fourier Series

We have already seen that the Fourier transform is important. For an LTI system, $x(t) = e^{i\omega t}$, then the complex number determining the output $y(t) = H(f)e^{i2\pi f t}$ is given by the Fourier transform of the impulse response:

$$H(f) = \int_{-\infty}^{\infty} h(t)e^{-i2\pi f t}dt$$

Well what if we could write arbitrary inputs as superpositions of complex exponentials, i.e. via sums or integrals of the following kind:

$$x(t) = \sum_k X_k e^{i2\pi f_k t}$$

Then notice, outputs of LTI systems y(t) will always take the form

$$y(t) = \sum_k X_k H(f_k) e^{i2\pi f_k t}$$

This is the root of the Fourier series.

Proposition 1.1. *Let x(t) be period with period T, so that the frequencies* $f_k = \frac{k}{T} = kf_0$*, and*

$$x(t) = \textstyle\sum_k X_k e^{i2\pi\frac{k}{T}t} \text{ - SYNTHESIS EQUATION}$$

$$= \sum_k X_k e^{i2\pi k f_0 t}$$

Then, $x(t) = x(t \pm mT)$*, and*

$$X_k = \tfrac{1}{T}\textstyle\int_0^T x(t)e^{-i2\pi\frac{k}{T}t}dt \text{ - ANALYSIS EQUATION}$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-i2\pi k f_0 t}dt$$

Proof: Use the property that

$$\int_0^T e^{i2\pi\frac{(m-n)}{T}t}dt = T\delta[m-n]$$

Then we have

$$\int_0^T x(t)e^{-i2\pi\frac{m}{T}t}dt = \int_0^T \sum_k X_k e^{i2\pi\frac{k}{T}t} e^{-i2\pi\frac{m}{T}t}dt$$

$$= \sum_k X_k \int_0^T e^{i2\pi\frac{(k-m)}{T}t} dt$$

$$= \sum_k X_k T\delta[k-m]$$

OK, so how do we use this. Well, for periodic signals with period T, then we just have to evaluate the Fourier series coefficients X_k.

Example 1.1.

1. x(t)=constant, then X_0=constant and $X_k = 0, k \neq 0$ for any period T.
2. $x(t) = e^{i2\pi f_0 t}$, then $T = \frac{1}{f_0}$, $X_1 = 1$, $X_k = 0, k \neq 1$.
3. $x(t) = \cos(2\pi f_0 t)$, then $T = \frac{1}{f_0}$, $X_1 = X_{-1} = \frac{1}{2}$, $X_k = 0$, $k \neq \pm 1$.
4. $x(t) = \sin(2\pi f_0 t)$, then $T = \frac{1}{f_0}$, $X_1 = \frac{1}{2j}$, $X_{-1} = -\frac{1}{2j}$, $X_k = 0$, $k \neq \pm 1$.

1.2 Relationship of Fourier Series and Fourier Transform

So, Fourier series is for periodic signals. Fourier transform is for non-periodic signals. Let's examine and construct the Fourier transform by allowing the period of the periodic signals go to ∞, see what we get.

Let's define $\tilde{x}(t)$ to be the periodic version of x(t), where x(t) has finite support $x(t) = 0$, $|t| \geq T/2$. Thus, $\tilde{x}(t \pm mT) = x(t), t \in [-T/2, T/2]$

Definition 1.1. *Define the Fourier transform of x(t) to be*

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi f t} dt$$

Then we have the relationship between FT and FS.

Proposition 1.2.

$$\tilde{X}_k = \frac{1}{T} X(kf_0) \text{ where } f_0 = \frac{1}{T}$$

where

$$\tilde{x}(t) = \sum_x \tilde{X}_k e^{i2\pi k f_0 t}, \text{ where } f_0 = \frac{1}{T}$$

Example 1.2. *Let* $x(t) = 1$, $t \in [-A/2, A/2]$, *and 0 otherwise. Then*

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi f t} dt$$

$$= \int_{-A/2}^{A/2} e^{-i2\pi f t} dt$$

$$= \frac{e^{-i2\pi f t} \Big|_{-A/2}^{A/2}}{-i2\pi f}$$

$$= \frac{e^{-i\pi f A} - e^{i\pi f A}}{-i2\pi f}$$

$$= \frac{\sin(\pi f A)}{\pi f}$$

Let $\tilde{x}(t \pm mT) = x(t), t \in [-T/_2, T/_2]$. Then,

$\tilde{x}(t) = \sum_k \tilde{X}_k e^{i2\pi k f_0 t}$ where $f_0 = \frac{1}{T}$

$$\tilde{X}_k = \frac{1}{T} X(kf_0) = \frac{\sin(\pi k f_0 A)}{T\pi k f_0} = \frac{\sin(\pi \frac{k}{T} A)}{\pi k}$$

OK, so we see that the Fourier transform can be used to define the Fourier series. Now what we would like to do is understand how to represent the periodic signals when the period goes to infinity $T \to \infty$, so that we can have a synthesis pair. Let's remind ourselves that $\tilde{x}(t)$ is the periodic version of x(t), where x(t) has finite support $x(t) = 0, |t| \geq \frac{T}{2}$.

Proposition 1.3. *Let $\tilde{x}(t)$ be periodic with period T, and $x(t) = lim_{T\to\infty} \tilde{x}(t)$. Then*

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{i2\pi f t} df$$

To see this,

$$x(t) = \lim_{T\to\infty} \tilde{x}(t) = \lim_{T\to\infty} \sum_k \tilde{X}_k e^{i2\pi k f_0 t}$$

$$= \lim_{T\to\infty} \sum_k \frac{1}{T} X(kf_0) e^{i2\pi k f_0 t}$$

$$= \lim_{T\to\infty} \sum_k X(kf_0) e^{i2\pi k f_0 t}$$

$$= \lim_{f_0\to\infty} \sum_k X(kf_0) e^{i2\pi k f_0 t} f_0$$

$$= \int_{-\infty}^{\infty} X(f) e^{i2\pi ft} df$$