The unit sample and unit step

Let's examine some special signals, first in discrete time, then in continuous time.

Definition 1.1. *The discrete time unit step is given by*

$$u[n] = \begin{cases} 1, n \geq 0 \\ 0, n < 0 \end{cases}$$

The unit sample or impulse is defined as

$$\delta[n] = u[n] - u[n-1]$$

We notice that they are related via the sum relation

$$u[n] = \sum_{k=-\infty}^{n} \delta[k]$$

Notice the unit sample sifts signals

Proposition 1.1. *The unit sample has the "sampling property," picking off values of signals that it sums against:*

$$x[n] = \sum_{k} x[n-k]\delta[k]$$

This is true for all signals, implying we can derive various properties, the "summed" and "differenced" versions. Defining $x[n] = u[n]$, then

$$u[n] = \sum_{k=-\infty}^{n} \delta[k]$$

Now let's examine linearity and time invariance.

Example 1.1. *Define the system with input x[n] and output y[n]=nx[n]. This system is linear but not time invariant.*

To see linearity is straightforward, Take linear combinations of inputs and verify outputs are linear combinations. To see the system is not time invariant, define input $x_1[n] = \delta[n]$*, then output is* $y_1[n] = n\delta[n] = 0$ *for all n. Now shift the input,* $x_2[n] = \delta[n-1]$*. But for this* $y_2[1] = 1$.

Definition 1.2. *The continuous time version has a similar form:*

$$u(t) = \begin{cases} 1, t \geq 0 \\ 0, t < 0 \end{cases}$$

The differentiated for is the unit impulse:

$$\delta(t) = \dot{u}(t) = \lim_{\epsilon \to 0} \frac{1}{\epsilon}(u(t+\epsilon) - u(t-\epsilon))$$

Proposition 1.2. *The sifting property follows for smooth functions:*

$$x(t) = \int x(t-\tau)\delta(\tau)d\tau$$

To see this, we can argue loosely swapping the order of the limit according to

$$x(t) = \lim_{n \to 0} \frac{1}{\epsilon} \int x(t-\tau)\big(u(\tau+\epsilon) - u(\tau-\epsilon)\big)d\tau$$

$$= \lim_{n \to 0} \frac{1}{\epsilon} x(t-\tau)d\tau = x(t)$$

In particular, defining x(t)=u(t), then we see the integrated property follows

$$u(t) = \int_{\sigma=-\infty}^{t} \delta(\sigma)d\sigma$$

$$\delta(t) = \dot{u}(t)$$