Mortgage Modification and the Decision to Strategically Default: A Game Theoretic Approach

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While numerous and varied opinions abound, there remains much confusion as to why relatively few mortgages are modified at a time when the demand to modify is historically high. To better understand this complex issue, we build a game theoretic model to quantify a number of economic incentives and costs surrounding critical dimensions of the lender’s decision to modify a loan and the borrower’s decision to strategically default in an attempt to encourage such a modification. We mathematically demonstrate that it is rarely economically rational for lenders to modify loans. For the borrower, we find that their negative equity position, growth rate in home prices, and the probability the lender will exercise its legal right to recourse represent the top three strategic default determinants.

Keywords: strategic mortgage default, loan modification, game theory, agent-based modeling.
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Introduction

Various stakeholders across the U.S. economy continue to express frustration and mounting confusion as to why the residential real estate credit market remains stagnant long after the initial housing crisis of September 2008. Evidence of the continuing tightness within this market is readily observable by examining mortgage origination statistics. For example, the Mortgage Bankers Association (2012) reports that while total U.S. mortgage originations averaged over $3 trillion per year from 2002-2007, since 2008 they have failed to reach the $2 trillion threshold in any single year. While this decline in origination volume is clearly observable, a complete understanding of the underlying root causes is not. One emerging thematic question throughout the continuing analysis of this housing crisis is whether, and to what extent, issues of strategic default materially influence housing market outcomes. For example, Wyman (2010), FICO (2011), and Guiso, Sapienza, and Zingales (2013) all document that strategic mortgage default, the decision of the borrower to exercise his put option (stop paying his mortgage) even though he has the financial means to maintain his payments, continues to rise. Interestingly, at the same time, lenders appear to be unwilling to modify loans, a decision which has also caused much consternation among both policymakers and market participants.\textsuperscript{1} Despite such governmental program efforts as the Home Affordable Modification Program (HAMP) and Home Affordable Refinance Program (HARP), all efforts to date remain unsuccessful in stimulating the flow of funds through the mortgage markets and improving loan performance outcomes. In fact, a July 2013 SigTarp Report states that almost half of all HAMP loan modifications have re-defaulted.
The report further suggests the “Treasury should conduct in-depth analysis and research to
determine the causes of re-default of HAMP permanent mortgage modification and the
characteristics of loans or the homeowner that may be more at risk for re-default.”\(^2\)

Recognizing that the decision of the borrower to strategically default on his mortgage and the
lender’s decision of whether or not to modify a loan are extremely complex and inter-related, we
take an entirely new approach. Specifically, we construct a game theoretic model that identifies
both the economic and behavioral incentives for borrowers and lenders to act in their own best
interests. Past models have generally examined only the economic incentives of default,
implicitly assuming wealth is the only materially relevant incentive of the borrower.\(^3\) However,
more recent studies such as Seiler (2014a, 2014b), Guiso, Sapienza, and Zingales (2013), Seiler,
Collins and Fefferman (2013), and Seiler et al. (2012) document a myriad of emotional
considerations as well.

From a game theoretic perspective, we also make a contribution in that most game theory models
are designed in a very simplistic fashion (using, for example, a 2x2 normal-form design) in
which the reader is asked to accept the assumption that the over simplified game can be applied
to understand real world phenomenon. We take a different approach. Our game theoretic model
is far more complex, built based on a robust set of variables (and their relations) found in
previous studies to influence borrower and lender decision-making. Using a backward inductive
approach, the resulting subgame perfect equilibrium strategies we achieve are thus based on the
specific value inputs from past studies. To learn how sensitive our model is to the results from
these past studies, we then focus on the key inputs of our model and conduct a Latin Hypercube Sampling (LHS) sensitivity analysis.

The central contribution of this investigation is our finding that it is Pareto optimal for the lender to modify a mortgage only in rare instances. This result is surprising in the sense that many people fault lenders for not modifying more loans, but is likely not surprising to those in the lending industry since they appear to have reached the same conclusion as we do. Moreover, we find that borrowers who are willing to strategically default on their mortgage create an incentive for the lender to modify the loan. This result reinforces and extends the previous arguments of Huberman and Kahn (1988), who demonstrate that the threat of foreclosure may significantly influence contract renegotiations.

In an investigation of the variables borrowers consider when making the strategic default determination, we find the most influential of these is the probability that the lender will pursue recourse subsequent to the foreclosure. Secondary to the probability of recourse, we find the borrower’s negative equity position is also important to the strategic default decision. Finally, the expected future growth rate in home prices and the negative impact on the borrower’s credit score (making it both more difficult and expensive to obtain future credit) round out the leading considerations in the strategic default consideration process. From the lender’s standpoint, the time it takes to complete the foreclosure process, the amount by which the monthly payment will be reduced, and the direct cost to the lender to modify the loan are all significant determinants of this critically important decision.
Game Theoretic Model

This section describes the game theoretic model and its Nash Equilibrium solution. A graphical description of the model is given along with the mathematical equations used to arrive at a solution. The interactions between the mortgage lender and the strategically defaulting borrower are represented in a sequential, extensive-form, game theoretic model with each node representing the choices of each player over a monthly time step period. The game is repeated for the following months with contemporaneous financial information being updated from the previous month’s actions. The game is finite, so a node chain instinct will only contain a maximum of 600 months (50 years).

The game ends when either of these conditions is met: a) the borrower pays off the mortgage entirely, or b) the lender forecloses on the property. A foreclosure occurs when (1) the borrower strategically defaults, (2) the borrower is unable to pay the mortgage, or (3) the mortgage life surpasses 50 years. Different games are constructed by using different input variables, which will affect the payoffs obtained by the players. These games are then solved using backward induction to determine a Nash Equilibrium strategy for each of the players.

Subgame Perfect Equilibrium

A subgame perfect equilibrium is a strategy set which forms a Nash Equilibrium for every subgame of the game (Selten, 1965). It is well known that every finite extensive game has a subgame perfect equilibrium (Gibbons, 1992; Fudenberg and Tirole, 1991). Since the finite game derived in this paper is sequential, as opposed to simultaneous, and has complete information, the equilibrium derived from the backward induction algorithm forms a subgame perfect
equilibrium because every subgame is solved in a sequential game by this backward induction algorithm. It should be noted that though the subgame perfect equilibrium was found, the trembling hand perfect equilibrium was not necessarily found (Selten, 1975). Though the trembling hand perfect equilibrium might be more appropriate for this type of game due to its inclusion of player uncertainty, it is notoriously difficult to derive for a large sequential extensive-form game such as ours.

**Uniqueness of the Nash Equilibrium**

For all the different versions of the game that were solved using backward induction, only a single Nash Equilibrium was found. For there to be multiple Nash Equilibria in an extensive-form sequential game, it would mean that at least one decision-node had multiple actions which had exactly the same expected payoff to that player making the decision. This never occurred within our game due to the level of accuracy within Microsoft’s Visual Basic for Applications programming code (i.e., double precision data variables were used which have a precision of $10^{-324}$). Thus, there was always a distinction between the multiple action choices, and only a single Nash Equilibrium was found for each game.

**Representation of the Nash Equilibrium**

The Nash Equilibrium for the game is a strategy for each player which tells him which action to take at each of the possible decision nodes. The number of nodes over the 50 year (or 600 months) life of the mortgage is on the order of $2^{2600}$. To explicitly transcribe the Nash Equilibrium strategy for both players would be nearly impossible in the space restrictions of a journal. Instead, we represent the Nash Equilibrium by the features it displays through a
deterministic path through the game. These features include whether the mortgage is completely foreclosed upon and whether the mortgage was ever modified.

**Model**

In this section, we provide a brief description of the mechanics of the sequential foreclosure game used to investigate the impact of changing a lender’s foreclosure and modification policies (as well as recourse probability). The game represents the relationship between a single borrower and a mortgage lender, and is played over a number of months, starting when an underwater borrower initially defaults and ending when either the mortgage is paid off in its entirety or when the property is foreclosed. We first describe the model using prose before moving on to a mathematical description. Figure 1 diagrams a summary of the decisions and events that occur during each node within the model. A decision node is represented by a square where one of the two players (the borrower or the lender) must make a choice. Terminal nodes are represented by a hexagon, and represent a state where the game ends. Circles represent deterministic events, while a rhombus represents a test to check if an environmental condition has been met.

(Insert Figure 1 here)

The game is initialized at month \( t = 0 \) (Node 1). If the property has (positive) equity, then the borrower always pays his mortgage (Node 2). If, however, the property is underwater, the borrower can choose between either defaulting or paying his mortgage. If he decides to pay, he is also required to make up for any previously missed payments, fees, and interest. If he decides
to default, then there may be a consequence for him including whether the lender chooses to 
foreclosure on the property (Node 3a).

The decision by the borrower to default is determined by the policy he is following. In this game, 
we focus on a Nash Equilibrium policy as opposed to the myopic one. The policy determines 
what action each player takes at each decision point (represented as a decision node in the 
diagram). For example, borrower actions can include the decision to default, to continue paying 
the mortgage, or to cure a default. The only decision the lender can make is to modify the 
mortgage if the borrower is presently in default. For the Nash Equilibrium policy, the player 
chooses the action that will reward him with the highest expected terminal utility (this is his only 
decision variable). The expected terminal utility is the expected total utility the player achieves at 
the end of the game, which is discussed in the Utility Equation section.

Identifying which action leads to the highest terminal utility means that the expected terminal 
utility for each action must be determined. This is done using the standard backward induction 
approach to solving extensive form games. An extensive form game can be thought of as a tree-
like structure, formed of state nodes which grow from a single starting node, where the branches 
represent the players’ actions that move the game onto the next state node; branching in this 
manner will continue until a terminal state has been reached\textsuperscript{12}. The way backward induction 
works is to move through this game tree in reverse. That is, we start at all the terminal leaves 
simultaneously, and determine the expected utility at each point based on the knowledge of the 
future round expected utility (which would have been determined at a previous induction step in 
the algorithm). This process of determining the node’s utility continues until the starting node
has been reach. For each chance node, the expected utility is determined from the expected utility gained from the next game state, based on the probabilistic distribution of selecting those states. For each action node, the action that would result in the highest expected utility is chosen, thus that action node is assigned the highest expected utility. The result of a backward induction algorithm is a policy, for each player, which determines what he should do at each action node; the algorithm also determines the utility for each node in the tree based on this determined policy. This policy is a Nash Equilibrium.

Returning to the game mechanics, if the borrower does not pay his mortgage, the loan must be in default for at least ‘\(M_d\)’ months before the lender will consider foreclosing. Therefore, if ‘\(M_d\)’ months have not passed, the lender will not foreclose (Node 3a). If the loan has been continually in default for ‘\(M_d\)’ months the lender will foreclose. If foreclosure occurs, the game ends because the property has been foreclosed due to the borrower defaulting for ‘\(M_d\)’ months (Node 4a). If the property is not foreclosed upon, the lender has the option to modify the mortgage payments if they have not already done so (Node 5). The lender will only modify if the mortgage can still be paid off in full within the 50 year time limit.\(^{13}\) Note the lender’s decision to modify the mortgage happens after the borrower’s decision is made at each of the monthly time steps.

Even if the player does decide to pay his mortgage, various events can occur before the next month. The game randomly determines if there was an “external termination event.” For example, a dramatic income shock due to events such as divorce, change in employment status, prolonged or acute illness, or even death may make the borrower unable to continue to pay his mortgage (Node 3b).\(^{14}\) Alternatively, positive changes in either a borrower’s personal economic
situation, or evolving market conditions, may offer the opportunity for the borrower to exit the loan via a utility maximizing prepayment of the entire outstanding loan balance.\textsuperscript{15} If such an “external termination event” occurs (Node 4b), the game ends because the property has either been foreclosed or the loan has been paid off in full. We assume this type of termination only happens when the borrower is paying off his mortgage, as opposed to defaulting, to limit the influence of random events.

Once all decisions have been made by both the lender and borrower, the simulation progresses to the next month with changes made to the state variables: payments are made, the value of the house changes, etc. (Node 6). Before progressing to the next month, we perform a debt-owed test to determine if the mortgage has been fully paid off (Node 7). If the mortgage has been paid off in full, the game ends (Node 8). If the mortgage has not been paid off in full, the game loops back to Node 2 and continues to the next month in the game.

\textit{Model Assumptions}

There are several key assumptions made in developing the model to ensure tractability.\textsuperscript{16} These include:

\begin{itemize}
  \item If the borrower decides to self-cure, he needs to make up for all missed monthly payments, penalties, and interest in that month as well as that month’s regularly scheduled monthly payment.
  \item An “external termination event” means something has happened where the borrower is unable to pay the mortgage now and in the future (e.g., prolonged illness or death). If this happens, there is no possibility of self-cure.
  \item An “external termination event” cannot occur when the borrower has already decided to strategically default.
  \item There is a 50 year time limit on the mortgage from the start of the game. Initially we assume a 30-year, fixed rate mortgage, but loan modification will add to the life of the loan.
\end{itemize}
• The lender forecloses on a property if it has been in default for ‘M_d’ months.
• The borrower can default multiple times as long as any single default does not last more than ‘M_d’ months.
• There can only be one modification of the mortgage by the lender, and the lender cannot un-modify the loan.
• Only one interest rate is considered for modification of the monthly payments. If it is shown that this rate is infeasible (i.e., more money is still owed than this rate would pay off in a 50 year time limit) then the mortgage payments cannot be modified.
• A fixed percentage is lost on the property when sold as a foreclosure.

Mathematical Model

The previous section provides a prose description of the model. In this section, the mathematical formulae for the model are derived. The game begins with a borrower who has a debt of ‘D_0’ (∈ ℝ) on a property with a monthly interest of ‘λ_i’ (∈ [0, 1]) and ‘M_0’ months initially left on the mortgage. Using this information, we are able to determine the fixed monthly payments of ‘c_p’ (for principal and interest), by the standard formula provided here:

Initial monthly payments (FRM):

\[ c_p = D_0 \frac{\lambda_i (1 + \lambda_i)^{M_0 - 1}}{(1 + \lambda_i)^{M_0} - 1} \]

As time ‘t’ passes, the remaining debt (D_t ∈ ℝ⁺) will naturally amortize. The value of the property, ‘V_0’ (∈ ℝ⁺), will vary over time based on the monthly growth rate, ‘λ_v(i)’. Note this growth rate may be either positive or negative. Thus, a new ‘V_t’ (∈ ℝ⁺) will be observed each month using the following formula:

Value of the home:

\[ V_t = \prod_{i=1}^{t} (1 + \lambda_v(i)) \cdot V_0 \]
We assume there is a fixed annual growth rate ‘α’, for which several different values are used in our analysis. Since ‘α’ can be negative, a strict form of the monthly growth rate is given below:

**Monthly growth rate:**

\[ \lambda_v(i) = \frac{\alpha}{|\alpha|} \sqrt[12]{|\alpha|} \]

During the game, the property value remains independent of the decisions taken by the borrower and lender, except of course in the event of foreclosure, as previously discussed. The borrower’s monthly payments, however, are influenced by various factors (e.g., if the borrower defaults and if the lender modifies the mortgage).

**Defaulting**

Throughout the course of the game, the borrower might choose to default at certain months.

Borrower defaults are tracked using a vector, \( v \in \{0,1\}^{600} \), where an entry value of one indicates the borrower defaulted in that month. The game is initialized by the borrower defaulting in the first month. Defaulting has negative consequences for the borrower, one of which is a reduction in his credit score.\(^{19} \) We normalize this reduction in credit score rating to an arbitrary value of ‘\( c_{csb} \)’. Thus, the total loss to the borrower’s credit score due to defaults is given as:

**Total loss to credit score due to defaulting:**

\[ CS(t, v) = \sum_{i=1}^{t} c_{csb} \cdot v(i) \]

Though little data is available, we assume the loss due to credit score reduction associated with defaulting is much less than from costs associated with a foreclosure ‘\( c_{fb} \)’, that is ‘\( CS(t, v) \ll \)’
$c_{fb}$. The lender also has a cost associated with a borrower defaulting which is called the carrying cost, $'c_{carry}' (\in \mathbb{R}^+)$. The carrying cost represents the tax and insurance payments the bank needs to continue to pay to prevent the government from gaining a more senior claim on the collateralized asset, as well as foregone interest and higher capital costs resulting from having to fund the REO.20

Beyond not paying his mortgage payments, the borrower gains an additional benefit from defaulting. More specifically, without incurring the monthly cash outflow associated with the mortgage payment, he may be able to pay off other debts or make a large purchase with the saved money. We represent this benefit as a fraction, $'\lambda_r' (\in \mathbb{R}^+)$, of the normal monthly mortgage payment. The total benefit of the borrower defaulting is given later in this paper once other key variables have been introduced.

**Self-curing**

Self-curing of a loan occurs when a borrower pays back missed mortgage payments (including penalties and interest) owed to the lender after defaulting. Borrowers will rationally opt to pursue such a strategy when: 1) they have the financial resources to do so, and 2) their expected disutility of foreclosure exceeds the financial costs associated with becoming current on the loan. As part of this process, the lender will impose a monthly penalty denoted by $'c_{db}' (\in \mathbb{R}^+)$. Thus, the total fees paid are given by:

Default fees paid by borrower (assuming not defaulting at time $'t'$):

$$FP(t, v) = \sum_{i=1}^{t} c_{db} \cdot v(i)$$
Mortgage Modification

The lender has the option to offer a defaulting borrower a modification (reduction) of his mortgage payments. This mortgage modification is represented as a fractional reduction, \( \lambda_m \) (\( \in (0, 1) \)), to the monthly payment. For tractability, the game only allows the lender to modify the mortgage once, and restricts such modifications to term extensions. We use \( t_m \) (\( \in [0, 1, 2, \ldots, 599] \)) to track in which month this modification occurred as it is used in the payoff calculation.

The lender is not obligated to modify the mortgage payments, and will only do so if it is in their financial best interest (i.e., increases their utility) to allow such re-contracting. If they do not modify the payments, we assign \( t_m \) a value of zero. We employ an index function to identify if the mortgage has been modified before time ‘\( i \)’:

\[
I_{tm}(i) = \begin{cases} 
1 & \text{if } 0 < t_m \leq i \\
0 & \text{o/w} 
\end{cases}
\]

Using this index function, we are able to determine the total payments the borrower has made by time ‘\( t \)’, as well as the remaining debt. The loan interest rate is represented by \( \lambda_I \) (\( \in [0, 1] \)). Per our restrictions on the allowable form of loan modification, we note that after a modification the remaining term of the mortgage is increased.

Total Payments Paid by borrower (assuming \( v(t)=0 \); i.e., not currently in default):

\[
TP(t, t_m) = \sum_{i=1}^{t} c_p. (1 - \lambda_m)^{t_m(i)} = c_p. (t - t_m) + (1 - \lambda_m). c_p. t_m
\]

Remaining Debt owed:
\[
D_t(t_m, \nu) = \left(1 + \lambda_t(t)\right) \left(D_{t-1}(t_m, \nu) - \left(1 - \nu(t)\right) \cdot c_p \cdot (1 - \lambda_m)^{t_m(t)}\right) \\
- \left(1 - \nu(t)\right) \cdot \nu(t - 1) \cdot \left(\sum_{i=1}^{t-1} \left(\prod_{j=i}^{t-1} \nu(j)\right) \cdot c_p \cdot (1 - \lambda_m)^{t_m(i)}\right)
\]

There is an extra benefit to the borrower for having a reduced mortgage payment similar to the extra benefit to the borrower for defaulting (i.e., the borrower again has more free cash flow). This extra benefit is represented as a fraction of the reduction in the monthly payments by ‘\( \lambda_r \)’ (\( \in \mathbb{R}^+ \)), which is assumed to be the same benefit that is gained from defaulting on the mortgage.

There is also a cost to the lender to modify the mortgage, ‘\( c_{mc} \)’ (\( \in \mathbb{R}^+ \)). Consistent with Riddiough and Wyatt (1994) and Wang, Young and Zhou (2002), we note that while modifying a loan may increase the probability of receiving future cash flows from the borrower, it also invokes the moral hazard problem that other borrowers who do not need a modification will seek one simply as a way to lower their monthly payment. As a result, the net effect on profitability of a lender being more willing to modify a loan can be either positive or negative. These values allow us to determine the various benefits accruing to both borrowers and lenders within the model.

Benefit per month (beyond payment) of reduced mortgage payment:
\[
BM = (1 - \lambda_m) \cdot \lambda_r \cdot c_p
\]

Total extra benefit gained from months defaulted:
\[
BD(t, t_m, \nu) = \sum_{i=1}^{t} \lambda_r \cdot \nu(i) \cdot c_p \cdot (1 - \lambda_m)^{t_m(i)}
\]
Foreclosures

Once the borrower defaults on the mortgage, the lender will foreclose after ‘$M_d$’ months have passed. The model tests to see whether the property is foreclosed upon using the following equation:

Tests for foreclosure:

$$\prod_{i=0}^{M_d-1} v(t - i) = 1$$

There are various costs associated with foreclosing on a property to the borrower, ($c_b \in \mathbb{R}^+$), such as moving expenses and the loss of reputational capital. Additionally, the borrower incurs potential costs associated with the loss of use of the property.22 The borrower will also suffer a major reduction in his credit score, making it both more difficult and more expensive to borrow in the future, which is included in the costs above.

There are also non-trivial costs to the lender associated with processing a foreclosure, ($c_l \in \mathbb{R}^+$). The lender will recoup some of these losses by selling the foreclosed property. A foreclosed property is assumed to be sold at a discount, which we represent as a fractional loss in value ‘$\lambda_f$’ ($\in [0, 1]$). Previous empirical research finds this foreclosure discount ranges from as low as 4% to more than 20%.23 The lender also has the option to pursue the borrower for the deficiency amount. We assume a fixed probability of recourse, ‘$P_R$’ ($\in [0, 1]$), is used by the lender to reduce the complexity of the model. If recourse does occur, there will be an additional cost beyond the deficiency amount borne by the borrower of ‘$c_{rb}$’ ($\in \mathbb{R}^+$) to cover legal fees. Similarly, lenders will also incur legal fees, so there will further be a fixed cost to the lender, ‘$c_{rl}$’
Finally, we assume the lender imposes a fee on the borrower if recourse is taken. Thus, the expected total costs of recourse to borrowers and lenders may be summarized as follows:

\[
RC_b(t, t_m, v) = P_R \left( c_{rb} + M_d \cdot c_{carry} + \max(0, D_t(t_m, v) - (1 - \lambda_f) \cdot V_t) \right)
\]

\[
RC_l(t, t_m, v) = P_R \left( c_{rl} - M_d \cdot c_{carry} - \max(0, D_t(t_m, v) - (1 - \lambda_f) \cdot V_t) \right)
\]

**External Termination Event**

Strategically defaulting is not the only reason a property might be foreclosed upon, or a mortgage loan terminated prior to its maturity date. Economic default can occur due to extraneous income shocks due to such events as a prolonged illness, job loss, and even death. This type of foreclosure falls under the heading of an “external termination event.” Similarly, borrowers may choose to prepay their outstanding mortgage obligations in full given life changes or in response to positive economic opportunities. To represent these events, we introduce a probability that the mortgage would be terminated outside of the model’s traditional default choice parameters, ‘\(\mu_G\)’ \((\in [0, 1])\). The value of this probability is set to \(1 - \sqrt[360]{1 - 0.01}\), which represents a 1% chance of an external termination event over the normal life of a mortgage. The external termination event represents the only random element within the game tree (which is represented as a chance node). Though other probabilities have been described within this model, they are always dealt with through expected values.
Utility Equations

There are various final outcomes that may occur during the game: external termination event, foreclosure due to strategic default, mortgage is paid off in full as scheduled, and foreclosure due to mortgage lasting more than 50 years. These outcomes represent the end nodes of the game tree, and through backward induction, determine the returns/payoffs expected by the borrower and lender. Which of the two players is the final decision maker will be determined by the type of terminal event. For example, the lender always initiates a foreclosure and the borrower initiates the final payment. Since a foreclosure automatically happens after a predetermined number of months, the lender has no choice in initiating a foreclosure if those circumstances arise. Using the equations derived above, the end node payoffs can be derived for each of the possible outcome scenarios. We next turn to a derivation of these payoff functions for each of the possible terminal events. Details about the payoffs from the intermediate steps are given in Appendix B.

Utility if home foreclosed (but not due to an external termination event):

The borrower and lender will receive different payoffs due to their differential utility functions. Since this is not a zero-sum game, each player’s payoff must be derived in turn. The borrower has several factors that affect his payoff. First, he no longer has the debt (unpaid mortgage balance) on his property to worry about, but there will be a reduction to his credit score from both defaulting and the foreclosure itself. The borrower still obtains any benefit from a previous mortgage payment modification made before he defaulted, but additionally incurs the (present value [PV] adjusted) loss from all the mortgage payments he has made. Finally, there is a chance the lender will seek recourse of the deficiency judgment stemming from the foreclosure. This expected loss must also be included in the equation.
Borrower:

\[
\text{Unpaid Mortgage Balance + Benefit Defaulting + Carry Cost + Benefit Modification} \\
\text{− PV of Total Payments Made − Credit Score Loss − Cost Foreclosure} \\
\text{− Recourse Cost}
\]

\[
D_t(t_m, v) + BD(t, t_m, v) + M_d \cdot c_{\text{carry}} + BM \cdot I_t(t_m) \cdot \max(t - M_d - t_m, 0) - TP(t - M_d, t_m) \\
- FP(t - M_d, v) - CS(t, v) - c_{fb} - RC_b(t, t_m, v)
\]

The lender’s payoff utility function is slightly different\(^{24}\). Upon foreclosure, the lender is able to recoup some of their losses through selling the property. Moreover, they have already obtained the money from past payments the borrower made before defaulting (as he might have defaulted and self-cured previously). The lender is further affected by the costs of foreclosure and mortgage modification. Note that the recourse cost will be negative, and thus have a positive effect on the lender’s utility.

Lender:

\[
\text{Property Sold − Remaining Mortgage Balance − Carry Cost} \\
+ \text{PV of Total Payments Received − Cost Foreclosure − Recourse Cost} \\
− \text{Cost Modification}
\]

\[
(1 - \lambda_f) \cdot V_t - D_t(t_m, v) - M_d \cdot c_{\text{carry}} + TP(t - M_d, t_m) + FP(t - M_d, v) - c_{fb} - RC_i(t, t_m, v) \\
- c_{\text{mt}} \cdot I_t(t)
\]

Utility if mortgage eliminated due to an external termination event:

The payoffs to the players in an external termination event are similar to the payoffs associated with the non-external foreclosure event case, with two major exceptions: 1) the timing of the event, and 2) the potential for (positive) equity. Termination occurs ‘\(M_d\)’ months after an external foreclosure termination event occurs, as the property has to be defaulted upon before it is foreclosed. Thus, we set ‘\(v(i) = 1\)’ for ‘\(i = t, \ldots, t + M_d\)’. As the foreclosure was not the
borrower’s choice, there is a chance the property has positive equity, and as such, the borrower
would receive the difference from the sale of the property and the balance due the lender. When
a property forecloses due to an external termination event it is possible that the property is above
water (i.e., has positive equity). Thus, the property may well be sold for an amount greater than
what is owed to the bank (including fees). As a bank cannot make a profit on a foreclosure it
must return the extra cash to the borrower, constituting an Above Water Return. Similarly, if the
external termination event results from either a prepayment or home sale event, $M_d$ will equal
zero, and any surplus cash flows will be returned to the borrower in the form of an Above Water
Return.

**Borrower:**

\[
D_{t+M_d-1}(t_m, \nu) + \max \left( (1 - \lambda_f) \cdot V_{t+M_d-1}(t_m, \nu), 0 \right) + BD(t+M_d - 1, t_m, \nu) + M_d \cdot c_{\text{carry}} + BM \cdot I_{t_m}(t) \cdot \max(t - 1 - t_m, 0) - TP(t - 1, t_m) - FP(t - 1, \nu) - CS(t, \nu) - c_{fb} - RC_b(t+M_d - 1, t_m, \nu)
\]

**Lender:**

\[
(Property Sold - Above Water Return) - Remaining Mortgage Balance - Carry Cost + PV of Total Payments Received - Cost Foreclosure - Recourse Cost
\]

\[
\min \left( D_{t+M_d-1}(t_m, \nu), (1 - \lambda_f) \cdot V_{t+M_d-1}(t_m, \nu) - M_d \cdot c_{\text{carry}} + TP(t - 1, t_m) + FP(t - 1, \nu) - c_{fl} - R_C(t + M_d - 1, t_m, \nu) - c_{mt} \cdot I_{t_m}(t) \right)
\]

**Utility if property’s mortgage is paid-off as scheduled:**
If the mortgage is completely paid-off by the borrower, then a much simpler payoff function is derived. The borrower now owns the property outright, but incurs the costs he accrued throughout the life of the mortgage.

**Borrower:**

\[
V_t + BD(t, t_m, v) + BM.I_{t_m}(t).\max(t - t_m, 0) - TP(t, t_m) - FP(t, v) - CS(t, v)
\]

**Lender:**

\[
PV \text{ of Total Payments Received} - \text{Cost Modification}
\]

\[
TP(t, t_m) + FP(t, v) - c_{mi}.I_{t_m}(t)
\]

The game can also terminate due to the mortgage passing the 50 year mark.

This utility function also varies depending on whether or not the property owner decides to default on his last payment. The payoffs are similar to those given for an external termination event but are not presented here due to the rarity of this occurrence.

**Justification for Complexity**

Though the model appears complex on the surface, in reality it contains only the essential elements of the strategic default-loan modification dilemma. There is an expectation within the field of economics to produce as parsimonious a model as possible, in accordance with Occam’s razor (Hamilton, 1852). As a result, there is a tendency to include only the variables that are known to have an impact on the output of the model. We argue it is also important to demonstrate which variables do not have an impact on the model’s outcome if it is reasonable to
hypothesize the variable might have an impact. To this end, our model demonstrates an exploration of the variable space as opposed to a normative model typically associated with Game Theory. With these considerations in mind, we believe we have struck a balance between constructing a tractable model while retaining sufficient richness to address the extremely complex problem at hand.

**Simulation Results**

The model shown in Figure 1 was developed into a simulation using Microsoft’s Visual Basic for Application (VBA) programming language. The model effectively represents a game tree, which was constructed in the simulation. This game was solved using a standard game theoretic technique (i.e., backward induction) to determine a Nash Equilibrium.

(insert Table 1 here)

*Input Variables*

Though the model contains random elements, expected values are used for the random component in the backward induction\(^\text{25}\). Since each run produces a deterministic Nash Equilibrium solution, there is no need to repeat the runs for statistically significance. The results presented are from two distinct batch runs of the simulation. Each batch run varies combinations of the different input parameters, as listed in Table 1. The first batch run only varies two input variables: the probability of recourse, and the degree of negative equity; the default value is used for all the other input variables. The results from the 10,000 varied runs from this first batch are used to produce Figure 2. The second batch run varies all the input variables using Latin Hypercube Sampling (LHS), as discussed in detail below.
The subgame perfect equilibrium, a form of Nash Equilibrium, is determined by the standard backward induction approach for sequence games. As the game contains a chance node element, expected values are used to determine each borrower’s payoff. The subgame perfect equilibrium determines a policy for each borrower. As the chance nodes of the game lead to a single non-terminal state, the combined policies represent a single path through the game tree. While there is no guarantee of a single equilibrium, in all the runs conducted, no multiple equilibrium result was ever found.

(insert Figure 2 here)

The first batch of runs is completed to determine: 1) under what conditions the property owner would strategically default, and 2) when the lender would modify the mortgage. By plotting the boundary between strategically defaulting and paying off the mortgage, shown in Figure 2, we are able to see the relation between the probability of recourse variable and negative equity. This relation takes the form of the inverse function, approximated by $1 - 2.10^5 \times x^{-1}$, which is an asymptotic function. This means that as the negative equity increases, the borrower is more likely to strategically default. Moreover, as the probability of recourse increases, the borrower is less likely to exercise his put option. This is especially true when the probability of recourse reaches one, as this implies the borrower will eventually have to pay off the debt. Thus, a recourse probability of one acts as the asymptotic boundary for the graph.
Another point of interest is under what conditions the lender modifies the mortgage. We note this only occurs near the above described boundary. To investigate this phenomenon further, a second batch of runs is completed in which the cost of a mortgage modification for the lender is excessively high (i.e., the lender would never choose to modify because doing so is far too expensive). The results from this batch run are also plotted in Figure 2. These results produce a similar boundary to the original results, but are slightly higher, indicating there are more occurrences of the borrower strategically defaulting. The times when the lender would rationally modify the mortgage occur exclusively between these two boundaries. Thus, the lender would only modify if the strategic result from the game would have been the borrower strategically defaulting. In this sense, the lender would defensively modify simply to prevent a strategic default on the part of the borrower.

This finding represents the key contribution of our study. Specifically, we mathematically demonstrate one potential reason why lenders rarely modify loans. It is simply not in their best financial interest to do so. Moreover, borrowers who threaten to strategically default cause lenders to have a greater incentive to modify. This graph supports the economic and behavioral incentives underlying the observed real world actions that borrowers voluntarily stop paying their mortgage in the hopes of inducing a loan modification.

**Sensitivity Analysis**

Verification and validation are important parts of any Modeling and Simulation (M&S) program. Verification involves checking that the model has correctly been coded in the simulation program, whereas validation requires ensuring the model is an accurate representation of the real
world system under consideration. While there is no standard method for conducting validation (Sargent, 2010), sensitivity analysis is commonly done to better understand the model’s results. Sensitivity analysis examines the impact of varying the input parameters on the simulation output and whether these variations appear reasonable\textsuperscript{26}. We perform Latin Hypercube Sampling (a sensitivity analysis) on our simulation model using partial correlation coefficients (PCC), the results of which are presented in Table 2 (see the appendix for a complete discussion of this technique).

![Insert Table 2 here](image)

Table 2 reports the results from the full model containing all 17 variables, as well as a more parsimonious model where only the eight significant variables are subjected to LHS analysis. From the PCC analysis, we observe the most economically important determinant of whether or not it makes sense to strategically default is the probability that the lender will pursue the deficiency judgment. In the extreme, if a borrower knew for certain a lender would come after him, he would never intentionally default on the loan. Doing so would not only ruin his credit, but also result in him paying back not only the outstanding balance on the mortgage, but also penalties and interest.

The second most influential determinant of the decision to strategically default is the negative equity position of the borrower. Consistent with Guiso, Sapienza, and Zingales (2013), we find that negative equity is not only a necessary condition, but also a robust determinant of the strategic default decision. Third, a related deterministic variable in our model is the growth rate

\textsuperscript{26}
in home prices. As home prices increase, borrowers have an incentive to hold the asset as opposed to exercising their put option. Conversely, a borrower would tend to favor exercising his put option when home prices are falling. Finally, while not as economically meaningful, additional factors statistically associated with a borrower’s decision to strategically default include the remaining term of the mortgage, the borrower’s fixed cost of foreclosure, the foreclosure discount attached to the property’s anticipated selling price, and the value of any extra benefits gained via the increase in free cash flow created by the suspension of mortgage payments.

Conclusions
While capital has begun to flow more freely back into the post-financial crisis mortgage markets, homeowners and potential homebuyers still find it challenging at times to (re)finance. Despite unsustainably low long-term interest rates, this lack of unimpeded credit flow is frustrating policymakers, borrowers, and the general public alike. In the current investigation, we ignore the political rhetoric and focus on the economic reasons why lenders might not want to modify existing loans. At the same time, we feel compelled to simultaneously examine the borrower’s decision-making strategy.

The increased incidence of borrower strategic default is well-documented, as homeowners threaten to stop making their mortgage payments in hopes of inducing the lender to modify their loan. This dangerous financial game is becoming more and more common as the stigma attached to mortgage default has dissipated in recent years. To understand the complex dynamics of the borrower-lender relationship, we build a game theoretic model and attempt to capture the
economic benefits and costs associated with various actions as the life of the mortgage unfolds over time.

Our primary result is a clear demonstration that only a narrow band exists where it is optimal for the lender to modify a mortgage. This result is consistent with ongoing observations in the marketplace. In a more detailed examination of the most influential determinants of the decision to default, we learn that the negative equity position, growth rate in home prices, and the probability that the lender will pursue a deficiency judgment subsequent to a foreclosure are the most deterministic variables when considering strategic default. In sum, the observed lack of widespread mortgage modifications may well be a rational response to the economic incentives faced by lenders.
References


FICO, Predicting Strategic Default. April, 2011, white paper.


Figure 1. Flow Diagram of the Game Theoretic Model
This figure conveys the decision tree as it branches through time surrounding the lender’s decision whether or not to modify a loan and the borrower’s decision to continue to pay or strategically default on his mortgage.

The different shapes represent different node types within the diagram:

- **Circle**: indicates a deterministic event.
- **Square**: Represents a decision that needs to be made by the borrower or lender.
- **Rhombus**: indicates a test on the environment.
- **Hexagon**: indicates a terminal node.
Figure 2. Boundary Graph between Lender Modification and Borrower Strategic Default Decisions
This graph delineates between where the borrower will versus will not strategically default on his mortgage. It also shows the area over which the lender will decide to modify a loan. The area inside the narrow band conveys why loan modification is so rare.

Above the line means that Borrower will pay off mortgage

Beneath the line means Borrower will strategically default and be foreclosed on

- Interface between player paying off mortgage and strategically defaulting
- Modification offered by Bank
- Results when banks are not allowed to modify mortgage
Table 1. Initial Model Values and Sensitivity Range Values for each Input
This table reports the initial estimate values for each input as well as the lowest and highest values tested in the subsequent Latin Hypercube Sampling sensitivity analysis.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Startin</th>
<th></th>
<th>Latin Hypercube Sensitivity Analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>g</td>
<td></td>
<td>Value</td>
</tr>
<tr>
<td>Initial property value</td>
<td>$100,000</td>
<td>$500,000</td>
<td></td>
</tr>
<tr>
<td>Negative equity position</td>
<td>$100</td>
<td>$200,000</td>
<td></td>
</tr>
<tr>
<td>Monthly interest (yearly)</td>
<td>5%</td>
<td>0%</td>
<td>20%</td>
</tr>
<tr>
<td>Initial months left on mortgage</td>
<td>330</td>
<td>12</td>
<td>330</td>
</tr>
<tr>
<td>Annual property growth rate</td>
<td>1%</td>
<td>-10%</td>
<td>20%</td>
</tr>
<tr>
<td>Carrying cost</td>
<td>$500</td>
<td>$0</td>
<td>$2,000</td>
</tr>
<tr>
<td>% Decrease in payment due to modification</td>
<td>15%</td>
<td>0%</td>
<td>40%</td>
</tr>
<tr>
<td>Probability of external termination event (over 30 years)</td>
<td>1%</td>
<td>0%</td>
<td>10%</td>
</tr>
<tr>
<td>Time defaulting until lender foreclosures</td>
<td>3</td>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>Penalty from lender for self-curing</td>
<td>$100</td>
<td>$0</td>
<td>$1,000</td>
</tr>
<tr>
<td>Credit-score loss due to defaulting</td>
<td>$100</td>
<td>$0</td>
<td>$1,000</td>
</tr>
<tr>
<td>Probability of recourse</td>
<td>5%</td>
<td>0%</td>
<td>100%</td>
</tr>
<tr>
<td>Fixed cost of recourse for borrower</td>
<td>$20,000</td>
<td>$0</td>
<td>$40,000</td>
</tr>
<tr>
<td>Fixed cost of recourse for lender</td>
<td>$10,000</td>
<td>$0</td>
<td>$20,000</td>
</tr>
<tr>
<td>Fixed cost of foreclosure for borrower</td>
<td>$18,500</td>
<td>$0</td>
<td>$97,000</td>
</tr>
<tr>
<td>Fixed cost of foreclosure for lender</td>
<td>$48,500</td>
<td>$0</td>
<td>$82,000</td>
</tr>
<tr>
<td>% Loss sale price of foreclosed property</td>
<td>11%</td>
<td>0%</td>
<td>100%</td>
</tr>
<tr>
<td>% Benefit extra from reduced mortgage payments</td>
<td>5%</td>
<td>0%</td>
<td>100%</td>
</tr>
<tr>
<td>Cost to lender to modify mortgage</td>
<td>$7,000</td>
<td>-$5,000</td>
<td>$20,000</td>
</tr>
</tbody>
</table>
Table 2. Latin Hypercube Sampling Analysis
This table reports the Partial Correlation Coefficients emerging from the Latin Hypercube Sampling analysis for both the full model as well as a more parsimonious model employing only those eight model parameters found to be statistically significant under the full model specification.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Full Model</th>
<th>Parsimonious Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Negative equity position</td>
<td>0.379**</td>
<td>0.427**</td>
</tr>
<tr>
<td>Months left on mortgage</td>
<td>-0.067*</td>
<td>-0.091**</td>
</tr>
<tr>
<td>Annual property growth rate</td>
<td>-0.355**</td>
<td>-0.295**</td>
</tr>
<tr>
<td>Carrying cost</td>
<td>-0.001</td>
<td></td>
</tr>
<tr>
<td>% Decrease in payment due to modification</td>
<td>0.072*</td>
<td>0.002</td>
</tr>
<tr>
<td>Probability of external termination event (over 30 years)</td>
<td>0.011</td>
<td></td>
</tr>
<tr>
<td>Time after defaulting until lender foreclosures</td>
<td>0.003</td>
<td></td>
</tr>
<tr>
<td>Penalty from lender for self-curing</td>
<td>0.011</td>
<td></td>
</tr>
<tr>
<td>Credit score loss due to defaulting</td>
<td>0.014</td>
<td></td>
</tr>
<tr>
<td>Probability of recourse</td>
<td>-0.567**</td>
<td>-0.590**</td>
</tr>
<tr>
<td>Fixed cost of recourse for borrower</td>
<td>-0.027</td>
<td></td>
</tr>
<tr>
<td>Fixed cost of recourse for lender</td>
<td>-0.002</td>
<td></td>
</tr>
<tr>
<td>Fixed cost of foreclosure for borrower</td>
<td>-0.272**</td>
<td>-0.308**</td>
</tr>
<tr>
<td>Fixed cost of foreclosure for lender</td>
<td>-0.039</td>
<td></td>
</tr>
<tr>
<td>% Loss in sale price of foreclosed property</td>
<td>0.267**</td>
<td>0.250**</td>
</tr>
<tr>
<td>% Benefit extra from reduced mortgage payments</td>
<td>-0.218**</td>
<td>-0.190**</td>
</tr>
<tr>
<td>Cost to lender to modify the mortgage</td>
<td>-0.019</td>
<td></td>
</tr>
<tr>
<td>Number of Observations</td>
<td>1,000</td>
<td></td>
</tr>
</tbody>
</table>

* Significant at 95%, ** Significant at 99%
Appendix A - Latin Hypercube Sampling

When performing a sensitivity analysis on a large number of variables whose values can take on a wide range of numbers, the resulting combinations can quickly become unwieldy. For example, with only six variables, each of which can assume 50 different values, the result is approximately 16 billion combinations. Even with today’s vast computing power, clearly it is not feasible to conduct this many combinations. Instead, Latin Hypercube Sampling (LHS) is a technique that identifies a subset of these combinations that statistically reflects full coverage of all interconnected combinations (McKay, Beckman, and Conover, 1979) to arrive at the same result. We now turn to a statistical discussion of this technique.

We begin with ‘k’ input variables and ‘N’ values for each variable. Each input variable ‘X’ is given a probability density function ‘f(X)’ and a domain \([x_{\text{min}}, x_{\text{max}}]\). This domain is segmented into \(N\) equally probable disjointed segments, each of which have their own minimum ‘\(x_{\text{min}}^i\)’ and maximum ‘\(x_{\text{max}}^i\)’. Hence,

\[
\frac{1}{N} = \int_{x_{\text{min}}^i}^{x_{\text{max}}^i} f(x) \, dx, \quad \forall \, i \in \{1,2,\ldots,N\}
\]

Determining the values of ‘\(x_{\text{min}}^i\)’ and ‘\(x_{\text{max}}^i\)’ can occur iteratively if ‘\(x_{\text{min}}^1\)’. The following formulae are used:

\[
\begin{align*}
    x_{\text{min}}^i & = \inf \left( x : F(x) > 0 \right) \\
    x_{\text{max}}^i & = F^{-1} \left( F \left( x_{\text{min}}^i \right) + \frac{1}{N} \right), \quad \forall \, i \in \{1,2,\ldots,N\} \\
    x_{\text{min}}^{i+1} & = x_{\text{max}}^i, \quad \forall \, i \in \{1,2,\ldots,N-1\}
\end{align*}
\]
As cumulative distribution functions are used in this iteration, it only needs to be slightly adapted for use with discrete variables. Given any ‘i’, a new random variable ‘Xᵢ’ can be determined with a probability density function (PDF) represented by:

\[
f_i(x) = \begin{cases} 
N_i f(x) & \text{if } x \in [x_{min}^i, x_{max}^i] \\
0 & \text{otherwise}
\end{cases}
\]

This is a PDF described by:

\[
\int f_i(x) dx = 1
\]

For each input variable ‘X’, a single sample ‘xᵢ’ is taken from each of the ‘N’ new random variables ‘Xᵢ’.

To determine the tuples of input variables for the simulation runs, a single sample is randomly selected from the set \{x₁, x², ..., xⁿ\} for each of the input variables, and a tuple of input variable values is formed. This process is repeated ‘N’ times without replacement to generate the complete set of samples.

In the current investigation, the sample is determined as follows. For each of the input variables, a minimum and maximum was determined, as previously shown in Table 1. A uniform distribution was generated from these minimum and maximum values for each input variable, and the resulting range was then split into 1,000 equally probable sub-distributions. A sample value was then taken from this sub-distribution and randomly added to the values of the other
input parameters to produce a complete set of values. Each of the 1,000 input parameters was subsequently run in the simulation.

**Partial Correlation Coefficients (PCC)**

Given a set of ‘N’ paired samples from a jointly distributed random variable ‘R’ = {Rᵢ, Rⱼ}, the sample correlation between the random variables can be calculated using the following formula:

\[ c_{ij} = \frac{\sum_{r=1}^{N} (rᵢ - \muᵢ)(rⱼ - \muⱼ)}{\sqrt{\sum_{r=1}^{N} (rᵢ - \muᵢ)^2 \sum_{r=1}^{N} (rⱼ - \muⱼ)^2}}, \quad i, j = 1, 2, \ldots, K \]

Where \{r₁, r₂, \ldots, r₇\} and \{r₁, r₂, \ldots, r₇\} are the samples with means ‘\(\muᵢ\)’ and ‘\(\muⱼ\)’ respectively. When considering rank-ordered samples, a special case of the Pearson correlation coefficient is used – Spearman Rank-Order correlation.

When conducting a simulation, it is possible that another variable that is also varied over the sample of simulation runs could have an impact on the correlation of the two variables. For example, consider the following sample of tuples: {1, 2, 4}, {2, 3, 9}, {3, 5, 12}, {4, 7, 17}, {5, 10, 25} and {6, 12, 32}. The first variable appears to have a positive relationship with the second variable, but so does the third. The question becomes the extent to which the first and second variables are correlated in the absence of the effect of the third.

Kendall’s partial correlation coefficient (PCC) is a technique used to remove the effects of other such variables (Kendall, 1942). The following steps show how the PCC ‘\(τ_{ij}\)’ is calculated given ‘k’ variables:
1. Define symmetric matrix $C := [c_{ij}]$

2. $B := [b_{ij}] = C^{-1}$ (exists as lead diagonal of $C$ is all ones)

3. $\tau_j = \frac{-b_{ij}}{\sqrt{b_{ii}b_{jj}}}$
Appendix B - Monthly Rewards

This appendix breaks down the utility equations given within the paper from the overall return a player can expect over the life of the game (up to 50 years), to equations which relate to the monthly reward obtained by the borrower’s actions (e.g., payment, default, self-cure). For the players, we define return as the expected payoff and reward as the immediate payoff for a particular action (Sutton and Barto, 1998). A rational player will be concerned with returns and a myopic player will be concerned with reward. Since standard game theoretic methods were used in our analysis, all players were assumed to be completely rational. The main purpose of this appendix is to provide the reader with further insight into our utility equations. These rewards relate to the different action scenarios presented in Figure 1; which node they relate to is shown in brackets.

This appendix discusses, in turn, the different payoff outcomes from the players’ different actions. The returns from terminal events, like foreclosure or complete payoff of the mortgage, are already covered in the utility section of the paper, thus the only actions by the borrower considered here are to continue to pay the mortgage, default on the mortgage, and self-cure. The payoffs for the lender are also briefly discussed. The different parts of each reward equation are discussed in turn to provide the reader a deeper understanding of each. The future return for the borrower is represented as ‘R(t, t_m, v, D_t, V_t),’ which is abbreviated to ‘R(t),’ and it is included in all equations to aid understanding of the borrower’s actual decision.
Buyers Reward for Regularly Paying Mortgage (Figure 1 - node 2 – pays)

The scenario associated with this reward is when the borrower has been making regular mortgage payments and continues to do so. There are four parts to this reward: future return, payment amount, benefit from reduced payments (if appropriate) and other reward. The payment amount is either \( c_p \) or \( (1-\lambda_m)c_p \), if the mortgage has been modified; the index function \( I_{tm}(t) \) ensures the correct amount is represented in this reward since \( x^0=1 \). As the payments are made by the borrower, it is represented by a negative reward in the equation (given below). There is a benefit from reduced payment due to a modification, \( (1-\lambda_m)\lambda_rc_p \). This is also included in the equation using the \( I_{tm}(t) \) function. We include an extra constant \( \delta_1 \) to represent other rewards the borrower receives: benefit of permanent accommodation, positive feeling from reducing the mortgage, etc. Note that the benefit from paying the mortgage principal is intrinsically included in the return function, \( R(t) \), hence not included in \( \delta_1 \). The quantity \( \delta_1 \) was assumed to be trivial when compared to the other rewards and was thus ignored in the main analysis.

Combining all these factors together, the borrower’s reward function for continuing to pay back the mortgage is:

\[
R(t) - (1 - \lambda_m)I_{tm}(t)c_p + I_{tm}(t)(1 - \lambda_m)\lambda_rc_p + \delta_1
\]

The reward the lender receives is simply the payment amount of \( (1 - \lambda_m)I_{tm}(t)c_p \). The exact value to the lender depends on whether the lender has previously modified the mortgage or not (Figure 1 – node 5). Since the impact from modification is considered a one-time cost, \( c_{ml} \), to the lender, there are no further impacts from reduced monthly payments to the lender. Note that even though a modified mortgage results in lower payments, the borrower still has to pay off the
complete principal amount and interest. Since \( \lambda_r \in (0,1) \) and \( \delta_1' \) are assumed to be small, it might be assumed that the resultant reward for the borrower is negative because the borrower has to pay money out. This is true in the myopic case (why pay money now when you do not have to?) but not for the long-term strategic case because, ultimately, paying off the mortgage will result in ownership of the house. This potentially positive long-term return is represented in ‘\( R(t) \).’

**Buyers Reward for Defaulting (Figure 1 - node 2 – defaults)**

The next scenario considered is when the borrower decides to default. Since he defaulted there are no payments to be made. There is a benefit from not having to pay the mortgage beyond the cash value amount (e.g., ability to now make a large purchase without credit due to having extra cash plus interest earned). This extra benefit is represented in the second term of the equation which is effectively the monthly payment multiplied by ‘\( \lambda_r \).’ There is a credit score loss to the borrower represented by ‘\( c_{csb} \).’ The borrower will also need to pay additional interest on the amount of principal that would have been paid if he had not defaulted. This extra interest is represented by the third term of the equation which is the monthly payment multiplied by the monthly interest rate. As with the previous scenario, an extra constant is included to represent other rewards the borrower receives. However, these rewards will be different when the borrower is defaulting (e.g., removal of hassle of making the mortgage payment, and as such is represented by a different constant \( \delta_2 \)).

\[
R(t) + \lambda_r (1 - \lambda_m) t_m(t) c_p - c_{csb} - \lambda_1(t)(1 - \lambda_m) t_m(t) c_p + \delta_2
\]
It is possible the value for the equation is negative, however, this does not mean that the defaulting option will not be picked by the borrower because return gained by this option may be *less negative* than the other options (e.g., choosing the lesser of two evils). For example, even if the credit score loss is severe, the borrower may still default due to being severely underwater on the property which is unlikely to every recover. The lender’s return when the borrower defaults is simply ‘$-c_{carry}$’ by definition of carry cost.

**Buyers Reward for Self-cure (Figure 1 - node 2 – pays)**

The final scenario considered is when the borrower decides to self-cure the defaulted loan. This means that the borrower decides to make his normal monthly payment plus missed back payments plus any monthly penalty imposed by the lender. Thus, the first part of the reward equation is the same as for the normal monthly payment scenario.

$$R(t) - (1 - \lambda_m)\sum_{i=1}^{t-1} \prod_{j=i}^{t-1} v(j) (c_{carry} + c_{db} + (1 - \lambda_m)\sum_{i=1}^{t-1} \prod_{j=i}^{t-1} v(j))$$

The second part is a little more complex as it needs to include all the payments required for all the previous defaulting months from the borrower’s current string of defaulted months. By string of defaulting months we mean the maximum number of previous months for which the mortgage was continually not paid. If any of the months considered in the string were paid, that would under our assumptions, imply the borrower had previously self-cured and thus all previous months had also been paid. A logic operator is created by using a special combination of
summation and product with binary values of ‘y’; this logic operator ensures that only the current string of defaulting months are considered. This logic operators works because the product of ‘y(i)’ values is only non-zero when all the values are one (as opposed to zero), thus only the current string of defaulting months is considered as all others produce a zero multiplier. For each month in the defaulting string, payment must be made to cover the carry cost, penalty and missed monthly payment. Not surprisingly, the reward from this action will most likely be severely negative for the borrower. Thus, they will only choose this action if there is a large future benefit to doing so.

1 For example, first lien modifications comprise less than $750 million of the $10.6 billion in consumer relief provided to consumers thus far through the nationwide settlement in the wake of the robo-signing scandal. See Berry (2012) for additional details of the settlement and potential help available to underwater borrowers.


3 For a pair of notable exceptions, see Riddiough and Wyatt (1994) and Wang, Young and Zhou (2002).

4 “Negative equity,” or what is commonly referred to as being “underwater” simply means the outstanding loan balance exceeds the value of the home. For a further discussion see in Sun and Seiler (2013), MacDonald and Winson-Geideman (2012), Shin, Saginor, and Van Zandt (2011), Zahirovic-Herbert and Chaterjee (2011), Zarudu, Levitan, and Guan (2011), Plaut and Plaut (2010), Zhou and Haurin (2010), and Seiler et al. (2008).

5 Conceptually, the game also ends when the loan is prepaid. This possibility is implicitly included through our external termination node. As sample borrowers enter the game with negative equity, and thus are unlikely to be able to freely exercise this prepayment option, the bulk of our textual discussion focuses on foreclosure based termination events.

6 Complete information within Game Theory means that the players know the payoffs of the other players and have the capability of determining the outcome of the game (or expected value) based on this knowledge (this kind of player is often referred to as “Homo Economicus”). The assumption of complete information is the standard approach when applying game theory though it does make strong demands of the players’ mental abilities. The alternative would be to use an incomplete knowledge player, but by doing so would severely increase the complexity of an already complex game. It would also require assumptions about what information the player did not have.

7 The trembling hand prefect equilibrium takes into account that a player is slightly uncertain about his opponent’s payoff functions. It introduces epsilon variables to the game and thus would have increased the complexity to solve the game. This type of equilibrium is useful when having to select between several Nash Equilibria; however, since our game only contains one Nash Equilibrium this is not a consideration.

8 The path that is considered removes all random influence on the games results; it is the one where an external foreclosure event never occurs.

9 For tractability, we focus on micro-level relationships. Of course, in the real world, the moral hazard problem stemming from granting modifications can cause herding behavior from individuals seeking to take advantage of the system. While we recognize this as likely, modeling such events becomes unwieldy. Further, to the extent that granting modifications influences borrower incentives to strategically default, this omission should lead us to overestimate the probability lenders will choose to modify a given loan. As such, we view our modelling choice as
providing a conservative estimate of the difficulty in making loan modification decisions and programs profitable for lenders.

10 Note that the borrower always starts with negative equity.

11 If the borrower defaulted in the previous round and now decides to pay, this is called a self-cure. Self-cures have historically been observed when a borrower is able to find employment after an extended period of being out of work. In today’s environment of negative equity, the increased incidence of strategic default has resulted in a precipitous drop in the self-cure rate, as many borrowers are financially unmotivated (or unable) to self-cure.

12 In the one player case, such as that employed in the current model, this is called a decision tree.

13 We cap the number of years at 50 simply to place a limit on the number of computations that need to be performed within the model. We should note that in no cases did a mathematical solution take more than 50 years to be achieved. Hence, the ceiling is not of critical importance. Additionally, as a simplifying assumption, modifications are assumed to take the form of term extensions as opposed to principal and/or rate reductions. Such alternative forms of modification, by definition, result in losses to the lender. Thus, they would be less economically desirable from the lender’s perspective than modifications which enhance affordability via lower payments through term extensions. As such, our model results should represent an optimistic view of lender willingness to modify mortgage loans.

14 This type of liquidity constraint can also be broadened to encompass alternative dimensions of “irrational” default.

15 Given that all borrowers enter the game in a negative equity position, we view it as relatively unlikely that they will be able to qualify for a refinancing opportunity. Nevertheless, we note that such an event could occur.

16 The vast majority of these assumptions may be relaxed without materially influencing the model’s ultimate equilibrium state. That said, doing so greatly enhances the complexity of the model while adding little in the way of meaningful economic insight.


18 We appreciate that the market can go both up and down, but we model the scenarios separately simply as a way to reduce the number of variants in the model. We know there are countless variables at play. In any game theoretic model, simplifications have to be made along the way to make the model more tractable.

19 For additional discussion on the role of default costs in influencing borrower and/or lender decision making, see Giammarino (1989), Curry, Blalock, and Cole (1991), Brueckner (2000), Posey and Yavas (2001), Harrison, Noordewier, and Yavas (2004), and Ben-Shahar (2006).

20 As a general rule under Basel III, one-to-four family residential mortgages have risk-weights substantially lower than those applied to real estate owned (REO).

21 While in practice it is possible to modify a loan more than once, or to reduce principal and/or interest rate levels as opposed to extending the loan term, such possibilities add unnecessary complexity to our model as they do not fundamentally alter the equilibrium relations we are attempting to illustrate.

22 To the extent housing markets are both deep and commoditized (thus allowing the defaulting borrower to buy/rent an equivalent unit at approximately the same amenity and risk-adjusted cost, with borrowers viewing alternative structures as near perfect substitutes), these costs should be negligible. On the other hand, to the extent the primary residence has positive, location (or unit) specific utility (e.g., good neighbors, proximity to friends, unique amenities, etc.) that would be lost by moving, the psychological costs associated with the loss of use may be an important determinant of borrower decisions. Note: utility maximizing borrowers presumably chose their initial location for a reason, and thus, barring changing tastes, preferences, and/or market conditions, a forced relocation likely does reduce their expected utility, potentially in a non-trivial manner.


24 Note that we have not taken into account insurance and tax payments the borrower might have paid throughout these equation derivations.

25 Most of our point estimates are from documented sources such as Ghent and Kudlyak (2011) and Harrison and Seiler (2015) for variables such as lender default times. Credit score loss due to foreclosure is from FICO (2011). Clauretie and Daneshvary (2011) provide estimates for our costs to foreclose to the lender, while Seiler et al. (2012) and Seiler, Collins, and Fefferman (2013) estimate costs to foreclose by the borrower. Seiler (2015a) offers
the probability of lender recourse. Still, since these estimates can vary across both time and location, we use Latin Hypercube Sampling techniques to measure and test the sensitivity and stability of the model to variation in these input parameter values.

26 Even though many of our point estimates are from empirical studies, it is still worthwhile to perform a sensitivity analysis to learn if estimation errors are likely to alter our conclusions. Moreover, sensitivity analysis allows us to identify which variables are most important to the model.

27 See studies by Seiler et al. (2012), Guiso, Sapienza and Zingales (2013), and Seiler (2014a,b), and Seiler 2015 a~c).