580.439 Course Notes: Linear circuit theory and differential equations

Reading: **Koch, Ch. 1**; any text on linear signal and system theory can be consulted for more details.

These notes will review the basics of linear discrete-element modeling, which can be considered to have three components: 1) generating models for the individual components of systems; 2) combining those component models into network models to represent a system; and 3) solving the resulting model equations in specific cases of interest. Linear electrical circuits will be considered, because these are usually the basis for neural membrane models. These notes will be most useful to persons who have not had a course in electrical circuit theory. It is assumed that readers are familiar with solution methods for linear differential equations.

Modeling the components of electrical circuits

Electrical circuits are built up of individual components, each of which is a two-port, as illustrated at right. That is, components have two connections or ports through which they exchange electrical signals with the rest of the system. Such a component can be characterized by the relationship between the current flowing through the device (I) and the electrical potential difference across the device (V). Current is the flow of electrical charge, measured in amperes, equal to coulombs/second, and electrical potential is measured in volts. The voltage is the potential across the device, meaning, in this case, the potential at the top connection minus the potential at the bottom connection. By convention, the voltage arrowhead is at the positive end of the voltage difference. By convention, the current is positive when it flows in the direction indicated by the current arrows and current is positive when it flows through the component from the positive side of the voltage arrow, as in Fig. 1. For practical circuits, charge cannot be created or destroyed within the component, so the current into one connection is exactly equal to the current leaving the other connection, as shown. This assumption will always be made.

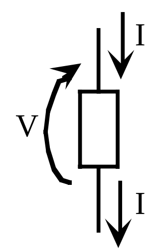
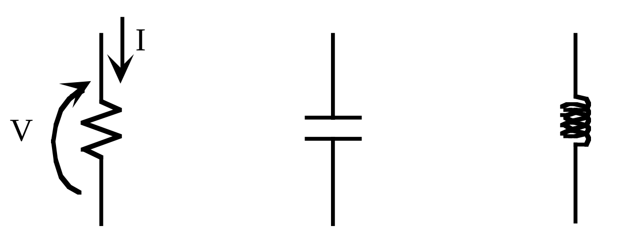


Fig. 1 Two port model for an electric circuit component (the box)

Analysis of electrical circuits is based on accurate models for the relationship between voltage and current in the two-port models of the components of the system (some electrical circuit components have more than two ports, requiring more complex models, but those objects are not of interest here). The three common circuit elements and their current-voltage models are shown in Fig. 2. In the resistor (at left), the current and voltage are proportional, with constant R, the resistance (units of ohms), which characterizes the device. This model is called Ohms Law and is also frequently written as $I=GV$, where the conductance G is $1/R$. The middle device is a capacitor, which stores charge on two parallel plates separated by an insulator. The voltage across a capacitor is related to the charge Q



$$V=IR \qquad I=C\frac{dV}{dt} \qquad V=L\frac{dI}{dt}$$

Fig. 2 Two port models for three common electrical circuit elements. From left to right they are a resistor, a capacitor, and an inductor. The definitions of current and voltage are the same for all three and are shown only for the resistor.

stored as $Q=CV$. The current I through the capacitor is equal to dQ/dt, and the two-port model of the capacitor given in Fig. 2 can be derived by differentiating. A capacitor is characterized by its capacitance C, in units of farads. The third model is for an inductor, a coil of wire, whose two port model is like that of a capacitor, except that it is current which is differentiated. The inductor is characterized by its inductance L, in units of henrys. For linear circuits, the parameters R, G, C, and L are constants. However in nerve membrane, the resistances (conductances) are not constant, and many of the interesting properties of neurons derive from the properties of these resistances.

Derivations of the two-port models from the physics of these components can be found in most undergraduate physics texts.

Question: The two-port models in Fig. 2 are given as differential equations. Consider how to write them as integral equations, i.e. write V as a function of I for the capacitor. To do this, you will have to introduce an initial condition on V. This topic will be important later.

The final two components that are important are ideal sources, a voltage source and a current source, shown in Fig. 3. The voltage source (left) produces a certain voltage V across its terminals, regardless of the amount of current it must provide. The current source (right) similarly forces a certain current I to flow through it, regardless of the rest of the circuit. These are obviously idealizations and real devices, like batteries, only approximate ideal sources. Additional components can be added in series or parallel with ideal sources to account for the limited ability of real-world sources to maintain a certain voltage or current.

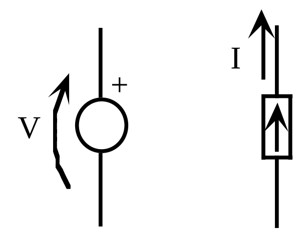


Fig. 3 Ideal sources: voltage source at left and current source at right.

Combining components into networks

Figure 4 shows three basic electrical networks. The principles that underlie solving for the various currents and voltages in these networks allow for networks of arbitrary complexity to be worked out. There are more powerful techniques than those presented here, which can be found in any electrical circuit theory book. However, for membrane models, it is not necessary to develop the subject of network theory in detail.

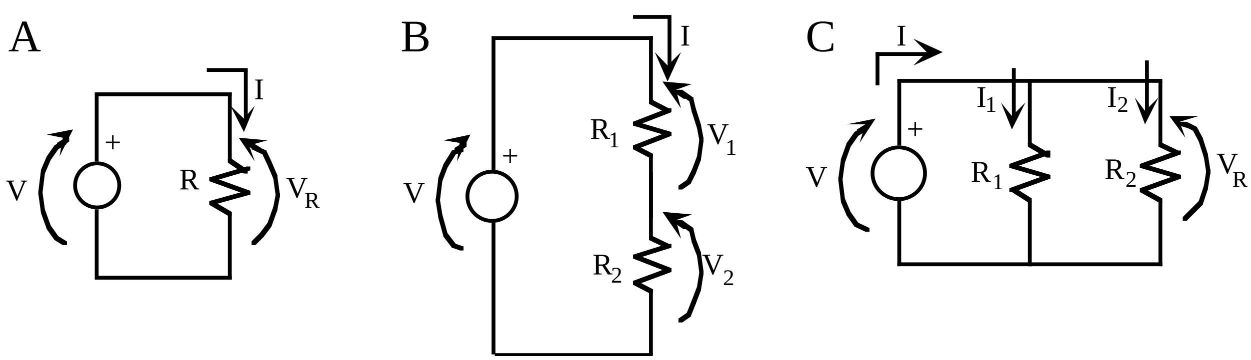


Fig. 4 Basic electrical circuits.

Figure 4A shows a simple circuit with a voltage source and a resistor connected in a series loop. There is a single current I which flows in a loop in the direction of the arrow through both the voltage source and the resistor. The voltage across the resistor is given by V_R=IR, from the two-port model in Fig. 2. Clearly the voltage across the current source V must equal the voltage across the resistor V_R, so that V=V_R=IR. From this equation, the current through the resistor is I=V/R. This is the first application of one of the two basic laws of circuits, Kirchoff's voltage law (KVL), or the loop law:

KVL: the sum of the voltages around any closed loop in a circuit is zero.

The transit around the loop must be made in one direction and the voltages must be added in a consistent way. In the case of Fig. 4A, going around the loop clockwise gives the result V-V_R=0. The signs on V and V_R must be opposite because the voltage arrows are transited in opposite directions. From this loop sum, we get the intuitively satisfying result that V=V_R, as used above. A final point is that it is assumed in circuit theory that there is no voltage drop across the connections between elements. That is, in transiting a loop, there are voltage changes only across the components of the circuit, not the connections between components.

Note that no account was made of the current through the voltage source. This is because an ideal voltage source always has a voltage V across its terminals, regardless of the current flow through it.

Figure 4B shows a series circuit, in which two resistors are placed in series with each other. Again there is only one current I and the currents flowing through the three components of the circuit are the same. Applying KVL and Ohms law gives:

$$V - V_1 - V_2 = 0 \qquad \mathrm{V}_1 = IR_1 \quad \text{and} \quad \mathrm{V}_2 = IR_2 \tag{1}$$

$$V = IR_1 + IR_2 = I(R_1 + R_2) \tag{2}$$

Note that Eqn. 2 is an expression of Ohms Law again, but with a resistor R_S=R_1+R_2. From this result, we can conclude that the circuits in Figs. 4A and 4B are essentially identical; two resistors in series, as in Fig. 4B, are equivalent to one resistor R_S whose resistance is equal to the sum of the two resistances.

Series resistances: two resistors in series are equivalent to a single resistor with resistance equal to the sum of the two individual resistances.

A series pair of resistors form a voltage divider, i.e. part of the voltage V appears across resistor R_1 in Fig. 4B and part appears across resistor R_2. Given the solution for current implicit in Eqn. 2, it is possible to write the following equations for the voltage across the two resistors:

Voltage divider relationship

$$V_1 = IR_1 = \frac{R_1}{R_1 + R_2} V \quad \text{and} \quad V_2 = IR_2 = \frac{R_2}{R_1 + R_2} V \tag{3}$$

Equation 3 gives the voltage divider relationships for the two resistors, which say simply that the voltage across one resistor in a series of resistors is equal to the same fraction of the voltage across the whole circuit as that resistor's resistance bears to the total resistance of the series.

Figure 4C shows a third situation, parallel combination of two resistors. Considering first the voltages in the network, it should be clear from KVL that $V=V_R$ and the voltages across the two resistors are equal, both V_R. This is done by applying KVL twice, once to the loop involving the voltage source and R_1 and once to the loop involving R_1 and R_2. There are now three currents I, I_1 and I_2. Based on the principle that current cannot be created or destroyed in connections among elements, it is clear that current I must divide into two parts, not necessarily equal, I_1 and I_2, and that $I=I_1+I_2$. This is a special case of Kirchoff's current law:

> **KCL**: The sum of the currents flowing into a node must be zero. Alternatively, the sum of the currents flowing out of a node must equal the sum of the currents flowing into the node.

A node is a point in the circuit at which connections among components meet. In the circuit of Fig. 4C there are two nodes, one at the top of the circuit and one at the bottom. In the circuit of Fig. 4B, there are three nodes, two between the voltage source and the two resistors (top and bottom) and one between the resistors. KCL applied to the circuit in Fig. 4B leads to the conclusion that the currents through all three elements are equal, as was assumed above.

Returning to the parallel circuit in Fig. 4C, the facts that the voltages across the components are equal (Eqn. 4) and the application of KCL to the top node (Eqn. 5) gives:

$$V = V_R = I_1 R_1 = I_2 R_2 \tag{4}$$

$$I = I_1 + I_2 = \frac{V}{R_1} + \frac{V}{R_2} = V\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \tag{5}$$

$$V = I\frac{R_1 R_2}{R_1 + R_2} \tag{6}$$

Equation 6 is, once again, an expression of Ohms law. Apparently the parallel combination of two resistors can also be expressed as a single resistance R_P, where

$$R_P = \frac{R_1 R_2}{R_1 + R_2}. \tag{7}$$

> **Parallel resistances:** two resistors R_1 and R_2 in parallel are equivalent to a single resistor, with the resistance given in Eqn. 7. Alternatively, two conductances G_1 and G_2 in parallel are equivalent to a single conductance equal to the sum of the individual conductances ($G_P=G_1+G_2$).

The rule stated above about conductances can be seen from Eqn. 5. Recalling that conductance G is equal to the inverse of resistance, Eqn. 5 can be rewritten as:

$$I = V(G_1 + G_2) \tag{5a}$$

from which the parallel conductance law follows. In parallel circuits, it is usually more convenient to solve problems using conductance, as opposed to resisitance.

A final point about Fig. 4C is the way the current divides between the two resistors. By substituting Eqn. 6 into Eqn. 4, the relationship between the currents in Fig. 4C can be solved as follows:

Current divider relationships

$$I_1 = \frac{V}{R_1} = \frac{1}{R_1} I \frac{R_1 R_2}{R_1 + R_2} = I \frac{R_2}{R_1 + R_2} \qquad (8)$$

$$\text{and similarly,} \quad I_2 = I \frac{R_1}{R_1 + R_2} \qquad (9)$$

The current divider relationships can be written in a form identical to the voltage divider relationships by converting resistances to conductances. Replacing R in Eqns. 8 and 9 with $1/G$ gives

$$I_1 = I \frac{G_1}{G_1 + G_2} \quad \text{and} \quad I_2 = I \frac{G_2}{G_1 + G_2} \qquad (10)$$

Question 1: Consider the circuit in Fig. 5. By working out successive series and parallel circuits and using voltage and current divider relationships, solve for current I_3 and voltage V_5 in terms of source voltage V and the resistances. The approach to solving such a problem is using KVL and KCL to write a linearly independent set of equations to solve. However, not all the equations resulting from KVL loops are independent and there is one fewer independent KCL equation than there are nodes in the network. There are formal methods for properly setting up these problems, which will not be considered here. There is an infinite variety of problems like this. Make some up yourself to practice.

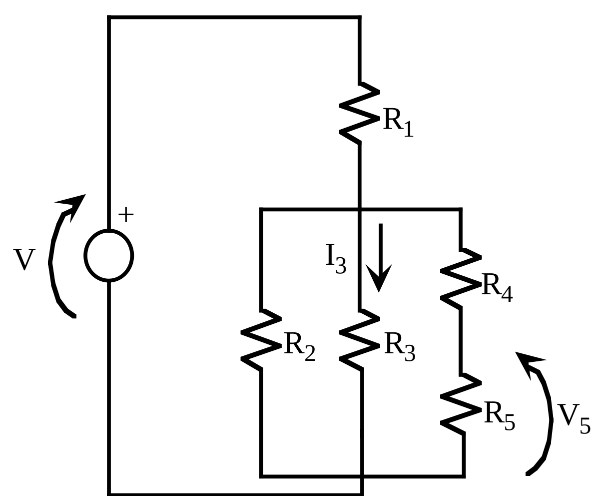


Fig. 5 Mixed series-parallel circuit .

Question 2: The circuit in Fig. 6 is often used to model the ionic currents through nerve membrane. What is the membrane potential V_m in terms of the individual ionic potentials V_{Na}, V_K, and V_{Cl}? Notice that the problem is stated in terms of conductances, not resistances, which is typical for nerve membrane models.

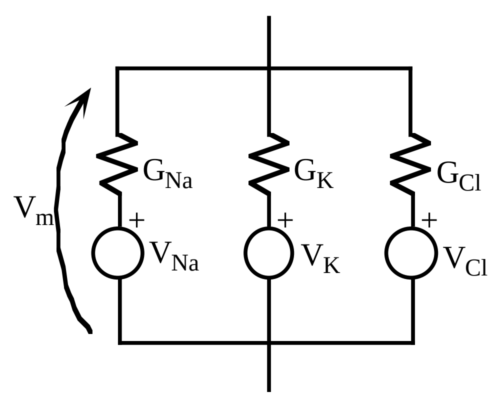


Fig. 6. Parallel combination of three sources. With the addition of a capacitor, this circuit is a model for nerve membrane.

A useful technique for solving problems like that posed in Question 2, when there are multiple sources in the network, is to solve the network for each source individually and then sum the solutions. For example, in the circuit in Fig. 6, one can solve for V_m with V_K=V_{Cl}=0, then solve for V_m again with V_{Na}=V_{Cl}=0, and then solve a third time with V_{Na}=V_K=0 and then add the three solutions. Note that setting a voltage source to zero means replacing it with a simple connection, a short circuit, which is an element with zero voltage change across it. This superposition principle works because circuit problems of the kind considered so far are linear, meaning that the solution will always be expressed as a weighted linear sum of the sources. The previous statement can be proved formally, but to see that it is true, consider that the solution to network problems means applying KVL and KCL to the nodes and loops in the circuit. Each of the resulting equations is a linear equation in terms of the source voltages and currents and the unknown voltages and currents. Thus network problems of the kind considered so far (with constant resistances) will always lead to simultaneous sets of linear equations and the solutions will be linear sums of the sources.

To apply the superposition technique when there are current sources in the system, the current sources that are to be set to zero are simply removed from the circuit, leaving an open circuit (i.e. a connection through which zero current flows).

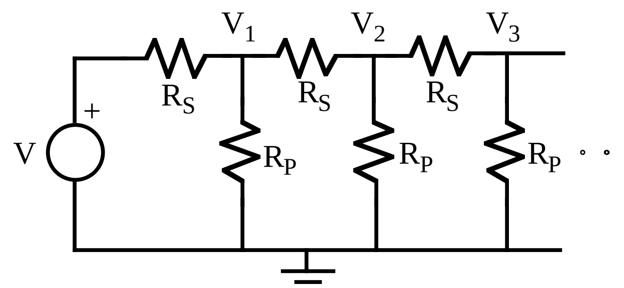


Fig. 7 First three sections of a semi-infinite ladder network.

Question 3: A ladder network is a semi-infinite circuit like the one drawn in Fig. 7. With the addition of capacitors, this circuit is used to model the dendrites of neurons, the so-called cable theory. This question demonstrates some interesting properties of ladder networks with resistor circuits. The system consists of series resistors R_S and parallel resistors R_P. The ladder is driven by a voltage source V at one end and is infinitely long at the other end. The problem is to show that the potential decays exponentially along the ladder, that is if V_n is the potential at the n^{th} node, the first three of which are labeled in Fig. 7, then $V_n = V a^n$, where a is the attenuation from node to node and V_0 is defined to be V. Potential arrows are not drawn in Fig. 7, but all potentials are assumed to be measured with respect to the node at the bottom of the circuit, the ground or reference node, indicated by the symbol ⏚. Show that the potential decays exponentially and compute a. Do this in two steps:

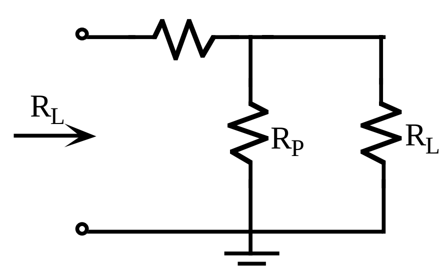


Fig. 7a The semi-infinite network can be replaced by a characteristic resistance R_L.

a) A ladder network has a characteristic resistance R_L which is the input resistance looking into the left end of the network (i.e. the whole ladder of resistors can be replaced by a single resistor R_L). Compute R_L. Hint: argue that the input resistance (i.e. the resistance looking into the left end of the finite network in Fig. 7a) should be R_L. Part of your argument should be that the original network is semi-infinite.

b) Using the result in a), compute a in terms of R_L, R_S, and R_P.

Source equivalents

A circuit containing resistors and sources can be replaced by an identical equivalent circuit, containing either one voltage source in series with one resistor (Thévenin equivalent) or one current source in parallel with one resistor (Norton equivalent). The circuits are equivalent in the sense that they behave exactly the same if connected in the same way to any external circuit. Two examples are shown in Fig. 8.

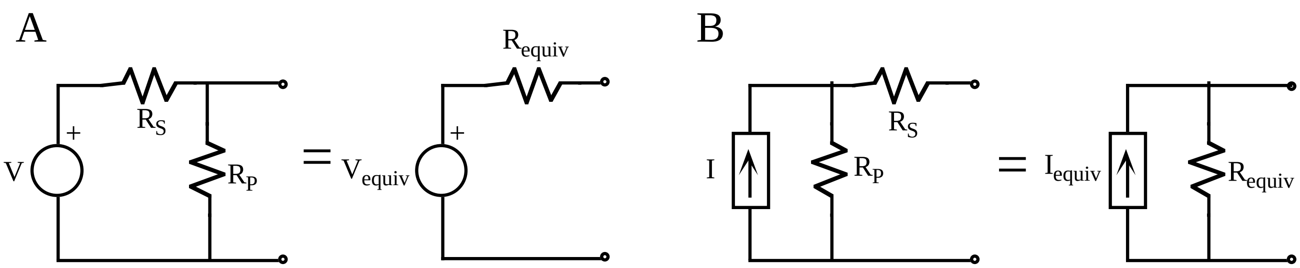


Fig. 8 A. Thévenin equivalent of a two-resistor circuit. B. Norton equivalent of another circuit.

In Fig. 8A, the two circuits are equivalent for any load (i.e. any circuit containing resistors and sources) connected between the terminals on the right (i.e. where the small circles are). Similarly, the two circuits in Fig. 8B are equivalent for any load connected between the terminals. The loads connected to the terminals can be any electrical circuit. By equivalent is meant that the voltages and currents in the external circuit would be the same, regardless of which of the two equivalent circuits it is connected to.

To find the Thévenin equivalent, proceed as follows:

> **Thévenin equivalent**: V_{equiv} is the open circuit voltage across the load terminals (i.e. with no load connected); R_{equiv} is the resistance presented at the load terminals with all sources in the circuit set to zero.

For the example in Fig. 8A, these rules give $V_{equiv} = V\,R_P/(R_S + R_P)$ and $R_{equiv} = R_S R_P/(R_S + R_P)$. For the Norton equivalent:

> **Norton equivalent:** I_{equiv} is the short circuit current obtained when the load terminals are shorted together. R_{equiv} is the resistance presented at the load terminals with all sources in the circuit set to zero.

For the example in Fig. 8B, these rules give $I_{equiv} = I\,R_P\,/\,(R_S + R_P)$ and $R_{equiv} = R_S + R_P$.

The proof of these rules is straightforward, but will not be given. However, you can easily verify that they work by trying a few simple examples. These equivalent circuits can often be used to simplify circuit problems considerably. The discussion above is stated in terms of resistors and sources, but the circuit can contain capacitors and inductors as well and the procedures are the same except that impedance (see below) is substituted for resistance.

Question 4: Derive the Norton and Thévenin equivalents of the circuits in Questions 1 (at the terminals of V_5) and 2 (at the terminals of V_m) and in Fig. 8. With regard to Fig. 8, what is the relationship between the Norton and Thévenin equivalents of the same circuit?

Question 4.5: One real-world approximation of an ideal voltage source is a battery. However, a battery is not a true voltage source in that the voltage at the battery terminals varies with the current through the battery. What is a model for a battery? Make a plot of the voltage at the battery terminals in the model, as a function of the current provided by the battery.

Differential equations for circuits with Ls and Cs

Real membrane circuits contain capacitors and (less often) inductors. For example, nerve membrane has a capacitance of about 1 μFd/cm^2, and a complete model of nerve membrane must have a capacitor in parallel with the ionic conductances of Fig. 6. The methods for dealing with networks of capacitors and inductors are the same as for resistors, except that the concept of resistance has to be extended slightly to accommodate the fact that the two-port models for inductors and capacitors are differential equations and not algebraic equations. For linear systems, the differential equations are usually solved by transforming them (Laplace or Fourier), so the equations end up algebraic in the end. As a result, the principles of network analysis discussed above apply essentially without change.

Figure 9 shows an example of a circuit containing a capacitor and two resistors. The goal is to solve for the voltage V_R in terms of the other parameters of the circuit. The two-port rules for the capacitor and resistors plus KVL give

$$I_C = C\frac{dV_R}{dt}\ , \quad I_R = \frac{1}{R_2}V_R \quad \text{and} \quad I = \frac{1}{R_1}(V - V_R) \tag{11}$$

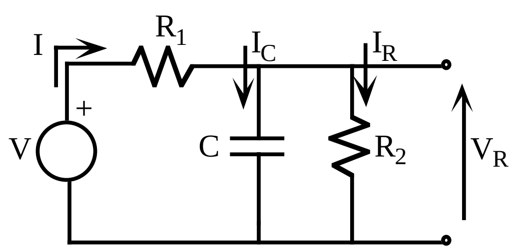


Fig. 9 Circuit involving a capacitor.

KCL at the junction of the components gives

$$I = I_C + I_R = C\frac{dV_R}{dt} + \frac{1}{R_2}V_R \tag{12}$$

Eliminating I between Eqns. 11 and 12 gives

$$I = \frac{1}{R_1}(V - V_R) = C\frac{dV_R}{dt} + \frac{1}{R_2}V_R \tag{13}$$

$$\frac{dV_R}{dt} + \left(\frac{1}{R_1 C} + \frac{1}{R_2 C}\right)V_R = \frac{1}{R_1 C}V \tag{14}$$

Equation 14 is a differential equation which relates the unknown voltage V_R to the input voltage V. Notice that this equation is linear in the voltage variables. Consideration of the process that was used to derive Eqn. 14 suggests that this will always be so. That is, the procedure for setting up the equations in a circuit problem consists of applying KCL, KVL, and the two-port equations for the components of the circuit. Each of these procedures will always produce an equation that is linear in the voltages and currents (i.e. terms like V^2, VI, I^2, $1/V$, $1/I$ etc. will not occur). Thus the solution of a circuit problem will always involve eliminating unknown voltages and currents among simultaneous sets of linear differential equations, which will always give a differential equation that is linear in the voltages and currents.

Initial conditions

To obtain a solution to a differential equation like Eqn. 14 requires boundary conditions; for a first order equation like this, one boundary condition must be specified. In circuit theory (and neural modeling) problems, it is usually appropriate to specify an initial value, i.e. a value of one of the variables at some time t_0. The solution is then the response of the system for time $t>t_0$. In Eqn. 14 it is the initial value of V_R which must be specified. This turns out to be a general rule for systems containing capacitors. The differential equations will always require that the initial voltages across the capacitors be specified. Similarly, for circuits with inductors, it is the initial current through the inductor that must be specified. To understand why initial values for these variables must be set, consider that the voltage across a capacitor is the time integral of the current through the capacitor, i.e. if V is the voltage across a capacitor C and I is the current through it, then

$$V(t) = V(0) + \frac{1}{C}\int_0^t I\,dt \qquad (15)$$

Equation 15 follows from the two-port model for the capacitor in Fig. 2. This equation is merely a way of saying that the voltage across a capacitor is proportional to the charge stored in the capacitor, since the charge is the time integral of the current through the capacitor (refer to the discussion of Fig. 2). Now, because it is the result of an integration, V cannot change instantaneously in time, assuming that I is finite; that is, $V(t)$ cannot have any discontinuities, because the properties of integration imply that the integral cannot have a step change if the integrand is finite. Moreover, the value of V at time t must depend on $V(0)$, i.e. on the initial value of V, as in Eqn. 15. This is equivalent to saying that the circuit adds or subtracts charge from the amount of charge stored in the capacitor initially and the total charge after a period of addition and subtraction depends on how much charge was there initially. Variables with this property are called state variables. State variables always require initial conditions in the equations for a system.

> **Initial conditions:** the initial conditions on linear circuit problems will always reduce to specifying the voltages across capacitors and current through inductors.

Solutions to circuit differential equations

The solution to Eqn. 14 can be obtained by several methods. For the problems typical of network theory, the values of the of the state variables are specified at some time t_0 and the solution is computed for time $t>t_0$. For example, suppose that $V_R(0)=V_0$. To simplify the notation, rewrite Eqn. 14 as

$$\frac{dV_R}{dt}+\frac{1}{\tau_{12}}V_R=\frac{1}{R_1C}V \quad \text{where} \quad \tau_{12}=\frac{1}{\frac{1}{R_1C}+\frac{1}{R_2C}}=\frac{R_1R_2}{R_1+R_2}C \tag{16}$$

The term τ_{12} is the product of a resistance and a capacitance; it has units of seconds and is called a time constant. The solution to Eqn. 16 can be found by multiplying both sides of the equation by the integrating factor $\exp(t/\tau_{12})$

$$e^{t/\tau_{12}}\left[\frac{dV_R}{dt}+\frac{1}{\tau_{12}}V_R\right]=\frac{1}{R_1C}V e^{t/\tau_{12}} \tag{17}$$

$$\frac{d}{dt}\left[e^{t/\tau_{12}}V_R\right]=\frac{1}{R_1C}V e^{t/\tau_{12}} \tag{18}$$

Now the equation can be integrated from the initial time (t=0) to time t giving

$$\int_0^t \frac{d}{du}\left[e^{u/\tau_{12}}V_R(u)\right]du=\int_0^t \frac{1}{R_1C}V(u)e^{u/\tau_{12}}du \tag{19}$$

$$e^{t/\tau_{12}}V_R(t)-V_R(0)=\int_0^t \frac{1}{R_1C}V(u)e^{u/\tau_{12}}ds \tag{20}$$

$$V_R(t)=V_R(0)e^{-t/\tau_{12}}+e^{-t/\tau_{12}}\int_0^t \frac{1}{R_1C}V(u)e^{u/\tau_{12}}du \tag{21}$$

An instructive case occurs when $V=V_1$ a constant. Then

$$V_R(t)=V_R(0)e^{-t/\tau_{12}}+\frac{\tau_{12}}{R_1C}V_1(1-e^{-t/\tau_{12}}) \tag{22}$$

$$V_R(t)=V_R(0)e^{-t/\tau_{12}}+\frac{R_2}{R_1+R_2}V_1(1-e^{-t/\tau_{12}}) \tag{23}$$

The voltage V_R across the capacitor and R_2 moves smoothly from the initial condition ($V_R(0)$) to the final value $V_1R_2/(R_1+R_2)$, as shown in Fig. 10. The transition occurs as an exponential decay with time constant τ_{12}; that is, the contribution of the initial value $V_R(0)$ decays exponentially to zero (first term in Eqn. 23) and the contribution of the steady state voltage increases exponentially (sescond term in Eqn. 23). Consider first the steady state value where dV_R/dt=0; this occurs for times large compared to τ_{12}. In this case there will be no current through the capacitor, from the capacitor's two-port model, so the circuit behaves as if the capacitor is not present. If the capacitor is removed from the circuit in Fig. 9, then the potential V_R is

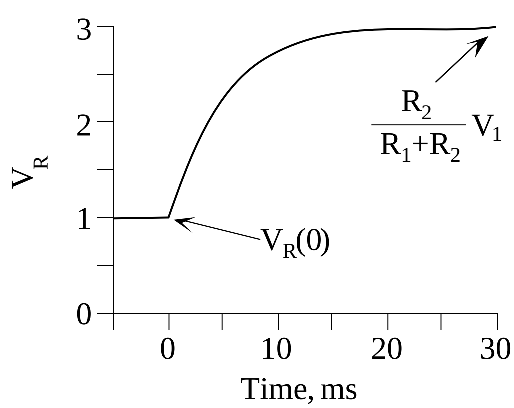


Fig. 10 Waveform of the voltage change of Eqn. 23 for the case where $V_R(0)$=1, steady state voltage is 3, and the time constant is 5 ms.

just $V_1R_2/(R_1+R_2)$, from the voltage divider relationship. What happens with the capacitor present is that current is driven into the capacitor to charge it up from $V_R(0)$ to potential $V_1R_2/(R_1+R_2)$, which occurs exponentially with time constant τ_{12}. A circuit consisting of one capacitor and a group of resistors and sources will always behave in this way, i.e. the potential across the capacitor will change exponentially in time from its initial value to a final value set by the resistors and sources. The time constant τ_{12} is a fundamental characteristic of such circuits and will always be equal to the product of the capacitance and the resistance seen from the capacitor's terminals with the sources set to zero.

The circuit of Fig. 9 is a simple model for the membrane properties of a neuron. The capacitor represents the membrane capacitance, resistor R_2 represents the resting resistance of the channels in the membrane, and the voltage source and resistor R_1 represent some external source applied to the membrane. Charging curves like Fig. 10 are seen in neurons when a step curent is injected into the cell body. The time constant is an important property of the cell which will be discussed in more detail later in the course.

Question 5: Consider the circuit in Fig. 11 which models a nerve membrane with capacitance C and resting conductance G. The resting potential of the cell is represented by the source E_R. The particular arrangement of G and E_R will be explained in a later set of notes. The current source I_{ext} represents a current injected into the cell through an electrode. Compute the membrane potential V_m with I_{ext}=0 and initial condition $V_m(0)$=0. Then compute the membrane potential as a function of time with $I_{ext}=I_0$, a constant, and $V_m(0)$ equal to the steady state value found with I_{ext}=0. This models injecting a step of current into the cell, which is the experiment usually done to determine membrane time constant.

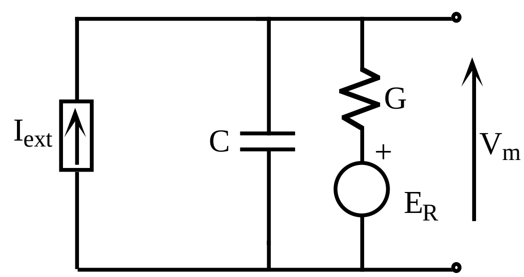


Fig. 11 Model of a nerve cell with external current injection.

Question 6: The circuit in Fig. 12 models a nerve membrane (G_R, E_R, and C) with a synapse attached (E_S, G_S). When the synapse is activated, the switch closes for about 1 ms, temporarily connecting the synapse circuit to the membrane circuit. First compute the steady state membrane potentials V_m with the switch open and closed. Second, compute the membrane potential versus time for a 1 ms switch closure, assuming that the membrane potential starts at E_R, i.e. that $V_m(0) = E_R$. (Be careful here, 1 ms is considerably shorter than the membrane time constant with switch open or closed).

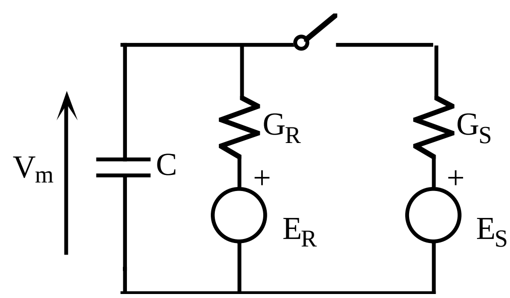


Fig. 12 Model of a nerve cell with a synapse. Closing the switch models activating the synapse.

Transformed differential equations, complex impedance

For linear systems with constant coefficients (i.e. R, G, C, and L that are constants, not dependent on time, voltage or current) it is usually most convenient to solve differential equations like Eqn. 16 using the Laplace transform. This method is appropriate for initial value problems of the type considered above. The Laplace transform is defined as follows:

$$\overline{V}(s) = \mathbf{L}[V(t)] = \int_0^\infty V(t) e^{-st} dt \tag{24}$$

$V(t)$ is a function of time, say a voltage in a circuit, and $\overline{V}(s)$ is its Laplace transform. The transform is a function of the complex number s, called the Laplace transform variable. An important property of the transform is that it is unique and invertable, that is, given a transform $\overline{V}(s)$, it is always possible to find the corresponding time function. The useful properties of Laplace transforms derive from their effect on differentiation. Consider the Laplace transform of dV/dt:

$$\mathbf{L}\left[\frac{dV}{dt}\right] = \int_0^\infty \frac{dV}{dt} e^{-st} dt = \left[V(t)e^{-st}\right] \Big|_{t=0}^{t=\infty} + \int_0^\infty V s e^{-st} dt \tag{25}$$

$$= -V(0) + s\overline{V}(s) \tag{26}$$

Equation 25 results from applying integration by parts and Eqn. 26 follows from the assumption, necessary to the existence of the Laplace transform integral in Eqn. 24 and to the invertability of the transform, that Re[s]>0, so that $e^{-st}V(t)$ evaluated as t approaches ∞ is zero. This result shows that taking a derivative in the time domain is equivalent to multiplying by s in the Laplace domain (and subtracting the initial value). This process can be extended to higher derivatives, giving the general rule

$$\mathbf{L}\left[\frac{d^n V}{dt^n}\right] = s^n \overline{V}(s) - \sum_{k=0}^{n-1} s^{n-1-k} \left.\frac{d^k V}{dt^k}\right|_{t=0} \tag{27}$$

Because of the derivative properties in Eqns. 26 and 27, the Laplace transform reduces a differential equation to an algebraic equation in s. The initial conditions of the problem are also incorporated into the algebraic equation. Using the example above and applying the Laplace transform to both sides of Eqn. 16 gives Eqn. 28:

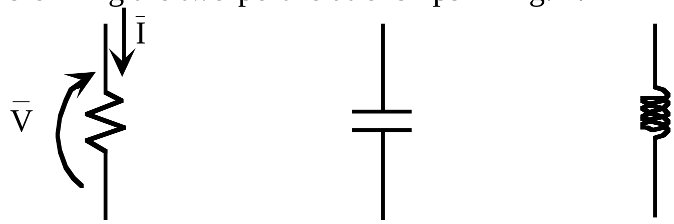
$$s\overline{V}_R - V_R(0) + \frac{1}{\tau_{12}} \overline{V}_R = \frac{1}{R_1 C} \overline{V} \tag{28}$$

$$\overline{V}_R(s) = \frac{1}{s + 1/\tau_{12}} V_R(0) + \frac{1/R_1 C}{s + 1/\tau_{12}} \overline{V}(s) \tag{29}$$

Equation 29 is the Laplace transform of the desired solution function $V_R(t)$, written in terms of the Laplace transform of the input voltage $\overline{V}(s)$ and the initial condition $V_R(0)$. The time function for V_R(t) can be recovered from Eqn. 29 using standard methods of transform inversion, which will not be covered here. In the case of a constant input, as in Eqns. 22 and 23, $\overline{V}(s) = V_1/s$ and the solution is the same as Eqn. 23.

The real advantage of using the Laplace transform technique is that one can avoid entirely deriving differential equations like Eqn. 16 by Laplace transforming the two-port models of the circuit components directly. Figure 13 shows the two-port models in the Laplace transform domain. These are obtained by Laplace transforming the two-port relationships in Fig. 2.

Fig. 13 Two port models for circuit components, in the Laplace transform domain.



$\bar{V}(s) = \bar{I}(s)\,R$ $\quad$ $\bar{V}(s) - \frac{1}{s}V(0) = \frac{1}{sC}\bar{I}(s)$ $\quad$ $\bar{V}(s) = sL\,\bar{I}(s) - L\,I(0)$

Ignoring the initial conditions (i.e. assuming that they are zero), these are all Ohms-law type statements. In each case, the two-port model takes the form $\bar{V} = \bar{I}\,Z(s)$, where $Z(s)$ is R for a resistor, $1/sC$ for a capacitor, and sL for an inductor. Z is called the impedance of the element and is a function of the Laplace transform variable s. Impedance is a generalization of resistance for networks containing capacitors and inductors; the inverse of impedance, analogous to conductance, is admittance, usually denoted $Y(s)$. Network models can be derived for circuits containing capacitors and inductors using the impedance relationships in Fig. 13 in exactly the same way as was done for resistors in the first few sections of these notes. The result will be exactly the same as the result of deriving the differential equation first in the time domain (as was done for Eqn. 16) and then Laplace transforming it.

Question 7: Derive Eqn. 16 entirely in the Laplace domain, beginning with the two port models in Fig. 13 and verify that the result is the same.

Question 8: Find the Thévenin equivalent of the circuit in Fig. 9 (at the terminals of V_R). For this, use a generalized definition of Thévenin equivalent in which there is a source and a series impedance (not resistance). Express both the source and impedance in the Laplace domain.

Question 9: The initial conditions can be considered to be voltage or current sources in the Laplace-domain two-port models. Convince yourself that the initial condition for the capacitor is equivalent to a voltage source in series with the capacitor with a value of $V(0)/s$ in the Laplace domain, or $V(0)u(t)$ in the time domain. ($u(t)$ is the unit step, equal to 0 for $t<0$ and equal to 1 for $t>0$. Its Laplace transform is $1/s$, which follows directly from Eqn. 24.) Similarly the initial condition for the inductor is equivalent to a voltage source in series with the inductor with value $LI(0)$ in the Laplace domain or $LI(0)\delta(t)$, where $\delta(t)$ is the impulse, equal to the time derivative of $u(t)$, in the time domain. The Laplace transform of $\delta(t)$ is 1.

Question 10: Show that initial conditions can also be represented as current sources in parallel with the circuit elements.

Question 11: Make a basic table of Laplace transforms by working out the transform of the following signals using Eqn. 24.

a) $u(t)$, the unit step, defined as

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases} \tag{30}$$

b) δ(t), the impulse or Dirac delta function. Informally, the impulse can be thought of as the derivative of $u(t)$. More formally, the impulse can be defined as the limit of a function, like a square pulse, whose amplitude is $1/T$ and whose width is T as T approaches 0. Note that the area under the function stays constant at 1. For the present purposes, it is sufficient to say that $\delta(t)=0$ for $t\neq 0$ and

$$\int_{-\infty}^{\infty} f(u)\delta(u)\,du = f(0) \tag{31}$$

(Hint: the impulse is assumed to be included in the integral in Eqn. 24.)

c) $e^{-at}u(t)$

d) $t\,u(t)$

e) $(1-e^{-at})u(t)/a$. (Hint: you should be able to derive this one from a) and c).)

f) $te^{-at}u(t)$

g) $\sin(\omega t)\,u(t)$ (Hint: remember that $\sin(\omega t) = (e^{j\omega\tau}-e^{-j\omega\tau})/2$, for $j=-1^{1/2}$)

h) $e^{-at}\sin(\omega t)\,u(t)$ (Same hint)

As a final example, consider the circuit in Fig. 14. This circuit comes up in a model for subthreshold oscillations in nerve membrane which will be considered later in the course. Proceeding as above using the two-port impedance models and KCL for the top node gives:

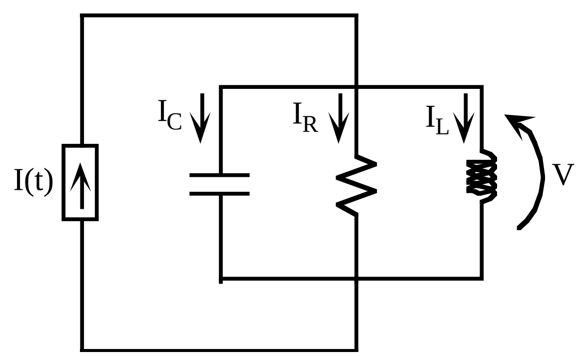


Fig. 14 Second order circuit consisting of a parallel combination of three circuit elements driven by a current source

$$\bar{I}_C = sC\bar{V} \qquad \bar{I}_R = \frac{\bar{V}}{R} \qquad \bar{I}_L = \frac{\bar{V}}{sL} \tag{32}$$

$$\bar{I} = \bar{I}_C + \bar{I}_R + \bar{I}_L = sC\bar{V} + \frac{\bar{V}}{R} + \frac{\bar{V}}{sL} \tag{33}$$

$$\bar{V} = \frac{\bar{I}}{sC + \frac{1}{R} + \frac{1}{sL}} = \bar{I}\,\frac{s/C}{s^2 + s/RC + 1/LC} \tag{34}$$

Equation 32 gives the two-port models for the components in the Laplace domain, assuming zero initial conditions. Equation 33 expresses KCL at the top node in the circuit. Equation 34 is a rearrangement in Ohms Law form expressing the Laplace transform of the voltage in terms of the Laplace transform of the current and the impedance of the circuit. It is instructive to try to derive the differential equation corresponding to Eqn. 34 in the time domain; the Laplace method is much easier.

To illustrate the standard method of inverting a Laplace transform, consider the response of the circuit in Fig. 14 to an impulse of current $I_0\delta(t)$. The Laplace transform of $I(\mathrm{t})$ is just I_0 and it is desired to find the voltage whose Laplace transform is as follows (from Eqn. 34):

$$\bar{V}(s) = \frac{I_0\, s/C}{s^2 + s/RC + 1/LC} \tag{34a}$$

The transform in Eqn. 34a is typical of transforms that arise in electrical circuits in that it is a ratio of polynomials in *s*. The usual strategy in inverting such transforms is to break them down into sums of terms of the form $a/(s+b)$, which is the transform of the exponential $a\mathrm{e}^{-bt}$ (see Question 11). The form of the inverse of Eqn. 34a depends on the roots of the polynomial in the denominator. If they are real, it will be a sum of exponentials. If they are complex, it will involve sines and cosines. It is instructive for this example to assume that the roots are complex. Rewrite Eqn. 34a in the following factored form:

$$\bar{V}(s) = \frac{I_0}{C}\,\frac{s}{s^2 + 2\delta s + \omega_0^2} = \frac{I_0}{C}\,\frac{s}{(s+\delta+j\omega_1)(s+\delta-j\omega_1)} \tag{35}$$

where $j=(-1)^{1/2}$, $\delta=1/(2RC)$ is called the damping coefficient and $\omega_0=1/(LC)^{1/2}$ is called the resonant frequency. The reason for these names will be apparent below. To make the factoring correct, $\omega_1^2 = \omega_0^2 - \delta^2$.

The transform can now be broken into two parts, a process called partial fraction expansion, as follows:

$$\bar{V}(s) = \frac{I_0}{C}\,\frac{1}{2j\omega_1}\left[\frac{\delta + j\omega_1}{s+\delta+j\omega_1} - \frac{\delta - j\omega_1}{s+\delta-j\omega_1}\right] \tag{36}$$

This decomposition is done by straightforward algebraic methods and can always be accomplished. You should verify that Eqns. 35 and 36 are the same. The goal is to express the transform as the sum of rational fractions with first or second order denominator polynomials, i.e. in terms of transforms whose inverses are known (as in Question 11). The inverse Laplace transform of $1/(s+a)$ is $e^{-at}u(t)$ (see Question 11), so the inverse transform of the right hand side of Eqn. 37 is the sum of two exponentials:

$$V(t)=\frac{I_0}{C}\frac{1}{2j\omega_1}\left[\left(\delta+j\omega_1\right)e^{-(\delta+j\omega_1)t}-\left(\delta-j\omega_1\right)e^{-(\delta-j\omega_1)t}\right]u(t) \tag{37}$$

which can be rearranged as follows:

$$V(t)=\frac{I_0}{C}e^{-\delta t}\left[\frac{\delta}{2j\omega_1}\left(e^{-j\omega_1 t}-e^{j\omega_1 t}\right)+\frac{j\omega_1}{2j\omega_1}\left(e^{-j\omega_1 t}+e^{j\omega_1 t}\right)\right]u(t) \tag{38}$$

$$=\frac{I_0}{C}e^{-\delta t}\left[-\frac{\delta}{\omega_1}\sin(\omega_1 t)+\cos(\omega_1 t)\right]u(t) \tag{39}$$

where use has been made of the following property of the complex exponential:

$$e^{j\omega t}=\cos(\omega t)+j\sin(\omega t) \tag{40}$$

Note that the final result in Eqn. 39 is entirely real, the imaginary components of previous expressions ultimately cancelled out. The solution in Eqn. 39 is a damped sinusoid, a sinusoidal term (in brackets) which oscillates at frequency ω_1 and decays exponentially with a time constant $1/\delta$. The voltage transient is plotted in Fig. 15, for a particular choice of component values. Note that the oscillation is at frequency ω_1 which is related to the parameters of the circuit by

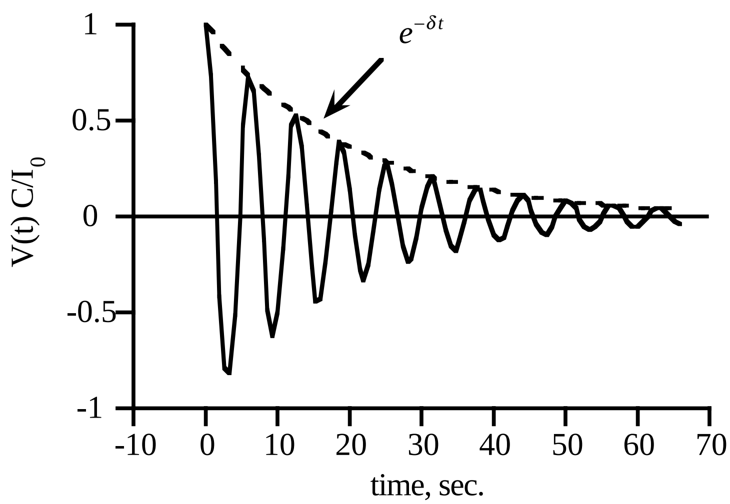


Fig. 15 Voltage transient across the parallel RLC circuit in Fig. 14 for an impulse of current. Solid line is the voltage, dashed line is the exponential envelope. Voltage plotted as VC/I_0. In this case, L=C=1 and R=10, so that δ=0.05, ω_0=1, and ω_1=0.998.

$$\omega_1=\sqrt{\omega_0^2-\delta^2}=\sqrt{\frac{1}{LC}-\frac{1}{4R^2C^2}}\approx\sqrt{\frac{1}{LC}}=\omega_0 \quad \text{if } \delta \ll \omega_0 \tag{41}$$

The approximation $\delta \ll \omega_0$ is true in resonant circuits, those in which the value of R is large. In this case, the circuit will "ring" or oscillate at its resonant frequency $\omega_1 \approx \omega_0$ for a long time if disturbed, as by an impulse in this example.

Question 12: Show using trigonometric identities that the final expression in Eqn. 39 can be expressed as (const)$e^{-\delta t}\cos(\omega_1 t+\phi)$, for a constant phase ϕ,

Question 13: Suppose the roots of the denominator polynomial in Eqn. 34a are real and negative. What will be the form of the inverse Laplace transform? (e.g. for R=0.33, L=1, and C=1.) Refer back to this question after you have read the last section below and studied Fig. 16.

Question 14: Invert the Laplace transform in Eqn. 29 using the method above and show that it gives the same solution as Eqn. 23 (Hint: $\overline{V}(s) = V_1 / s$ in this case.)

The sinusoidal steady state and frequency response

The system considered in Fig. 14 and Eqn. 34 can be analyzed another way which provides further insight into its properties. In this case, we assume that the input current is a sinusoidal function of time, say $I_0\cos(\omega t)$. The system is assumed to have been allowed to run for a long time so that transient responses to the turning-on of the current have died down. Such transient responses will take the form of Eqn. 39, which is the transient response to an impulse. The transients for the sinusoidal input case can be computed directly from the Laplace transform in Eqn. 34 by letting $\bar{I}(s)$ be equal to the Laplace transform of $I_0\cos(\omega t)u(t)$. The solution will be the sum of the transients, which will decay as $e^{-\delta t}$, and a steady-state sinusoidal component of the form $A\cos(\omega t+\phi)$. After a time long compared to $1/\delta$, only the sinusoidal component will survive. Thus, it is reasonable to assume that the steady-state voltage $V(t)$ will be a sinusoidal function of time at the same frequency ω as the excitation.

With the assumption of sinusoidal steady state, all signals in the circuit will take the form $A\cos(\omega t+\phi)$ where A is the amplitude and ϕ is the phase (or time shift) of the signal. Different signals will have different amplitudes and phases, and the goal of the analysis is to find these values. It is convenient to represent such signals in exponential form as

$$A\cos(\omega t+\phi) = A\left(\frac{e^{j(\omega t+\phi)} + e^{-j(\omega t+\phi)}}{2}\right) \tag{42}$$

which follows from Eqn. 40. In analyzing the sinusoidal steady state, it is sufficient to compute the responses of the circuit to the complex exponential $e^{j\omega t}$ since real signals can be computed from it using Eqn. 42.

The sinusoidal steady state has properties similar to those of the Laplace transform domain, in that differential equations are reduced to algebraic equations. For example, the two-port models for circuit components take the following form:

$$\text{resistor:} \quad V = IR \quad \text{becomes} \quad \bar{V} = \bar{I}R \tag{43}$$

$$\text{capacitor:} \quad C\frac{dV}{dt} = I \quad \text{becomes} \quad j\omega C\bar{V} = \bar{I} \tag{44}$$

$$\text{inductor:} \quad V = L\frac{dI}{dt} \quad \text{becomes} \quad \bar{V} = j\omega L\bar{I} \tag{45}$$

These equations follow because for any signal of the form $Ae^{j(\omega t+\phi)}$, differentiation with respect to t is the same as multiplying by $j\omega$. In the sinusoidal steady state, the term $j\omega$ has the same role as s in the Laplace domain. The only difference is that there are no initial conditions to deal with. As discussed above, the initial conditions are so far in the past that they have no effect. The two-port models in the sinusoidal steady state (Eqns. 43-45) again take the form of Ohms law relationships; they can all be written as $\bar{V} = \bar{I}\,Z(j\omega)$, where Z is the complex impedance, equal to R for a resistor, $1/j\omega C$ for a capacitor, and $j\omega L$ for an inductor.

The same methods that were used in Eqns. 32-34 can now be used again to derive the relationship between the sinusoidal steady state voltage and current in the circuit of Fig. 14. Because the two-port models take the same form, with s replaced by $j\omega$, the net result is simply to replace the s variables in Eqn. 34 with $j\omega$, giving

$$\bar{V} = \bar{I}\frac{j\omega/C}{-\omega^2 + j\omega/RC + 1/LC} \tag{46}$$

Equation 46 expresses the relationship between voltage and current in this circuit when the current excitation is sinusoidal. To see how this relationship is useful, assume that $\bar{I} = I_0 e^{j\omega t}$. The voltage can then be expressed as $\bar{V} = Ae^{j(\omega t+\phi)}$, where the unknown amplitude A and phase ϕ are obtained as follows:

$$\bar{V} = Ae^{j(\omega t+\phi)} = I_0 e^{j\omega t}\frac{j\omega/C}{-\omega^2 + j\omega/RC + 1/LC} \tag{47}$$

$$Ae^{j\phi} = I_0\frac{j\omega/C}{-\omega^2 + j\omega/RC + 1/LC} \tag{48}$$

$$A = I_0\left|\frac{j\omega/C}{-\omega^2 + j\omega/RC + 1/LC}\right| \quad \text{and} \quad \phi = \angle\left(\frac{j\omega/C}{-\omega^2 + j\omega/RC + 1/LC}\right) \tag{49}$$

Equation 47 is a restatement of Eqn. 46 with the exponential signals given explicitly. In Eqn. 48, the common component ($e^{j\omega t}$) has been cancelled leaving only the amplitude and phase of the voltage on the left hand side. Equation 49 provides expressions for the amplitude and phase of the voltage in terms of the amplitude and phase of the complex number on the right hand side of Eqn. 48. Equation 49 is the <u>frequency response</u> of the system. It describes, in this case, how the amplitude A and phase

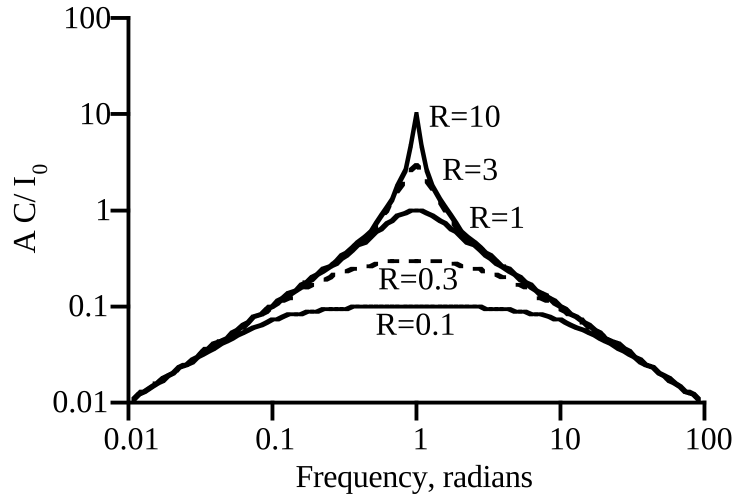


Fig. 16 Amplitude of the sinusoidal voltage across the parallel circuit of Fig. 14 versus radian frequency. $L=C=1$ as before and results are shown for five values of R, corresponding to damping δ=0.05, 0.17, 0.5, 1.7, and 5.

ϕ of the voltage in the system of Fig. 14 change as the frequency of the sinusoidal excitation current changes.

Figure 16 shows a plot of the amplitude of the frequency response of this system, i.e. a plot of $A\ C/I_0$ versus radian frequency ω. Again it is useful to recast the relevant expression in terms of ω_0 and δ, the parameters derived above. When this is done, the amplitude becomes

$$A = \frac{I_0}{C}\left|\frac{j\omega}{(\omega_0^2-\omega^2)+j\omega 2\delta}\right| = \frac{I_0}{C}\frac{\omega}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+4\omega^2\delta^2}} \tag{50}$$

At low frequencies, the amplitude goes to zero due to the ω term in the numerator of Eqn. 50. At high frequencies, the denominator becomes approximately ω^2 and the amplitude again goes to zero as $1/\omega$. In between there is a resonance, a frequency at which the response is a peak. When the damping δ is small (R large), the resonance occurs near $\omega=\omega_0=1$ radian, where the denominator of Eqn. 50 has a minimum. The amplitude when $\omega=\omega_0$ is $I_0/(C2\delta)$, so the amplitude is larger as the damping decreases. Note that the axes in Fig. 16 are plotted logarithmically, so the gain change is substantial. When the resistor is large, the circuit in Fig. 14 is a resonator, meaning that it gives a large response to currents at frequencies near 1 radian and smaller responses at other frequencies.

Question 15: Find the frequency response of the circuit in Fig. 9, for the ratio $\overline{V}_R/\overline{V}$. Plot it in a plot like Fig. 16 for the following values, typical of nerve membrane: C=1 μFd/cm^2, R_2=10 KΩ·cm^2, and R_1=10 R_2.

Question 16: The development of Eqns. 47-50 and Fig. 16 assumed that the signals in the system are complex exponentials $I = I_0 e^{j\omega t}$ and $V = Ae^{j(\omega t+\phi)}$. Show that if the signals are real, $I=I_0\cos(\omega t)$ and $V=A\cos(\omega t)$, the same solutions are obtained for A and ϕ. Hint: obtain solutions for $I = I_0 e^{j\omega t}$ and $I = I_0 e^{-j\omega t}$ and use the fact that the system is linear to compute the response to $I_0\cos(\omega t)$. (If the system is linear, then the response y to inputs x_1+x_2 is equal to the sum of the individual responses y_1 to x_1 and y_2 to x_2, that is $y=y_1+y_2$.)

This section contains only an introductory view of the subject of the frequency response of systems. The subject is much more extensive and allows analysis of signals and systems in many useful ways. In particular, the subject of the Fourier transform, which is closely related to the frequency response, has not been mentioned. However, the material presented above is sufficient for neural modeling and more detailed treatments of these issues can be found in any book on signal theory.