

580.439/639 Homework #9

Not to be handed in

Problem 1

Consider learning rules similar to the Perceptron Learning Rule (PLR) using a *linear* perceptron, in which $v = \vec{w} \cdot \vec{x}$. The bias term has been absorbed into the weight vector \vec{w} and there is no squashing function. Consider the problem of supervised learning to make this perceptron give responses $\{v^p | p = 1 \dots P\}$ to input patterns $\{\vec{u}^p | p = 1 \dots P\}$. The dimension of the weight and input vectors is N .

Part a) Define the *overlap matrix* $\mathbf{Q} = [q_{jk}]$ as $q_{jk} = \sum_{i=1}^N u_i^j u_i^k$, where u_j^i is the j^{th} component of the i^{th} pattern vector. Let the perceptron's weights be

$$\vec{w} = \sum_{j=1}^P \sum_{k=1}^P v^j (\mathbf{Q}^{-1})_{jk} \vec{u}^k,$$

where $(\mathbf{Q}^{-1})_{jk}$ means the jk^{th} element of the inverse of \mathbf{Q} . Write a matrix equation for \mathbf{Q} in terms of the $N \times P$ pattern matrix \mathbf{U} whose columns are the input patterns \vec{u}^p and show that this weight vector gives output v^l to input pattern \vec{u}^l .

Part b) Under what conditions on the input patterns does the analysis of part a) work?

Part c) Define the error in a linear perceptron as the difference between the output and the desired output summed across the input patterns:

$$E = \frac{1}{2} \sum_{j=1}^P (\vec{w} \cdot \vec{u}^j - v^j)^2$$

The weights could be found by gradient descent in which $\vec{w}_{\text{new}} = \vec{w}_{\text{old}} - \epsilon \vec{\nabla} E$ where ϵ is the learning rate. Show that this leads to a learning rule that looks the same as the PLR (except for the missing $\text{sgn}[\]$ functions).

Problem 2

Part a) Show that the function below behaves like a Lyapunov function for the Hopfield network discussed in class, in that H decreases monotonically when the network's state values change in time. S_j is the output value of the j^{th} neuron as usual.

$$H = -\frac{1}{2} \sum_{i,j} w_{ij} S_i S_j \quad (*)$$

where the summation is taken over all pairs of states. To do this, you will need to assume that $w_{ij} = w_{ji}$. Make it clear why this assumption is needed. Hint: consider $H' - H$, the change in the energy function when a single state flips, that is H changes to H' when S_k changes to S_k' and no other states

change. Show that $H'-H$ is always negative when S_k changes according to the discrete update rule for the Hopfield net:

$$S_k^! = \operatorname{sgn} \left[\sum_{j=1}^N w_{kj} S_j \right]$$

Have you proven that Eqn. (*) is a Lyapunov function, as defined in class?

Part b) To be a Lyapunov function, H must be minimized at an equilibrium point. We want the equilibrium points to be at patterns stored in the net. Suppose a Hopfield net has only one pattern $\vec{u} = [u_1, u_2, \dots, u_N]$ stored. Consider the function

$$H = -\frac{1}{2N} \left(\sum_{i=1}^N S_i u_i \right)^2 \quad (**)$$

Argue that this is minimized when $S_i = u_i$, i.e. the pattern is an equilibrium point. Is the minimum unique (i.e. can you find another equilibrium point)? Now use the definition of weights for the Hopfield net to show that (*) and (**) are the same. For the case of a net with many patterns stored, (**) could be summed over all the patterns; would the same result hold? Is the function (**) positive definite? If not, what can be done to make it so?