580.439/639 Final Exam Solutions, 2007

Problem 1

Part a) Larger conductance in the postsynaptic receptor array; action potentials (usually Ca^{++}) propagating in the forward direction in the dendrites; amplification by subthreshold conductances (Na^{+} or Ca^{++}). The increase in input impedance of the dendrites as the synapse moves away from the soma would also be accepted.

Part b) Theories of neural plasticity often postulate a Hebbian learning rule in which synapses are strengthened or weakened depending on the association of presynaptic activity (activation of the synapse) and postsynaptic activation of the neuron. Backpropagation of the AP allows the postsynaptic state of the neuron to be communicated to synapses throughout the dendritic tree.

Part c) Ca^{++} admitted to the spine head is confined to the spine and therefore serves as a specific signal associated with that spine, useful for plasticity. Depolarization of the spine head spreads with little attenuation to the dendrite and is thus a general signal for the dendritic tree.

Part d) In order for the decay to be exponential, the current must be applied to the end of a single long cable. If the equivalent cylinder theorem applies to this neuron, then current I_1 is applied to the end of that cylinder. The decay would be exponential to the extent that the equivalent cylinder is long. Current I_2 produces a complex decay that includes current into side branches and is unlikely to be exponential.

Part e) The equilibrium point for this system is

$$x_1 = \frac{1}{k} \qquad x_2 = 0$$

Using the hint, try the following candidate for Lyapunov

$$U(x_1, x_2) = a(x_1 - \frac{1}{k})^2 + (x_2 - 0)^2$$

This function is positive definite (assuming a>0) everywhere except at the equilibrium point where it is zero. Its time derivative is

$$\frac{dU}{dt} = \frac{\partial U}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial U}{\partial x_2}\frac{dx_2}{dt} = 2a(x_1 - \frac{1}{k})x_2 + 2x_2(1 - kx_1 - rx_2)$$

$$= (2a - 2k)x_1 x_2 - (2\frac{a}{k} - 2)x_2 - 2rx_2^2$$

If a=k, then $dU/dt = -2rx_2^2$ which is negative everywhere except at the equilibrium point. Thus U is a Lyapunov function on the whole plane and the equilibrium point is a stable attractor.

Part f) A linear perceptron can compute any classification that is linearly separable, meaning that $\vec{w}^T\vec{x}+b$ is >0 for one class and <0 for another , for some weight vector $\vec{w}$ and a bias b. For the separable examples, many sets $\{b,\vec{w}\}$ are possible.

(1) Separable {-1, 1, 1}

(2) Separable {1, 1, 1}

(3) Not separable

(4) Separable { 1, 1, -1}

(5) Not separable

Problem 2

Part a) Let G_m and C_m be the membrane conductance and capacitance per unit area and R_i be the axoplasmic resistivity. Then the parameters (length constant λ and input conductance G_∞) of the cylinders are given by the following:

$$\lambda_j = \sqrt{\frac{a_j}{2G_m R_i}} \qquad G_{\infty j} = \sqrt{\frac{2G_m}{R_i}}\,\pi\, a_j^{3/2} \qquad j = 1, 2, 3 \quad .$$

The conditions for the equivalent cylinder are then

$$\begin{array}{ll} L_j = l_j \big/ \lambda_j \quad j = 1, 2, 3 & L_1 + L_2 = L_1 + L_3 \ \ (\text{or } L_2 = L_3) \\ & G_{\infty 1} = G_{\infty 2} + G_{\infty 3} \end{array} .$$

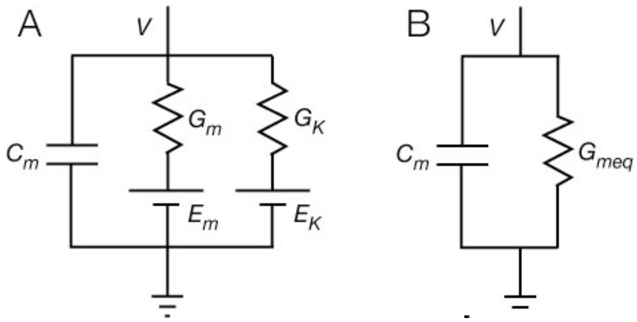
The third condition, matching loads, is met by the zero axial-current condition at the ends of the tree. Note that if the membrane parameters G_m, C_m, and R_i are assumed to be uniform throughout the tree, then they cancel from the equations above, leaving only

$$\begin{array}{rl} L_2 = L_3 \ \ \text{requires that} & l_2 \big/ \sqrt{a_2} = l_3 \big/ \sqrt{a_3} \\ G_{\infty 1} = G_{\infty 2} + G_{\infty 3} \ \ \text{requires that} & a_1^{3/2} = a_2^{3/2} + a_3^{3/2} \end{array} .$$

The properties of the equivalent cylinder are

$$L_{EC} = L_1 + L_2 \qquad \lambda_{EC} = \lambda_1 \qquad l_{EC} = \lambda_1 (L_1 + L_2) = l_1 + l_2 \frac{\lambda_1}{\lambda_2} \qquad a_{EC} = a_1 \qquad G_{\infty EC} = G_{\infty 1}$$

Part b) Adding the voltage-dependent channel just adds a battery/resistor pair G_K E_K in parallel with the circuit (C_m, G_m, E_m) for the resting membrane (going back to the first step in the derivation of the cable equation). Thus the resting membrane is represented by the circuit *A* in the figure at right. As long as G_K is a constant, which will be so if *V* is constant everywhere in the dendritic tree, then the batteries and resistors can be replaced by a new resting circuit consisting of Thevenin equivalents, as in circuit *B*, which holds if the zero of membrane potential is reset to E_{meq}, as is usually done in deriving the linear cable equation.



$$E_{meq} = \frac{G_K E_K + G_m E_m}{G_k + E_m} \quad \text{and} \quad G_{meq} = G_K + G_m$$

Now the parameters of the dendritic tree are changed by replacing G_m with G_{meq} in the equations of part a). Note that G_{meq}>G_m so the length constants will be shorter, and the G_∞s will be larger. $V_1 = E_{meq}$.

Problem 3

Part a) The current *I* produces potentials as follows

$$V_I = K_{II} I \quad V_j = K_{Ij} I \quad V_0 = K_{I0} I \ ,$$

and plugging these into the definition of subunit in the problem statement gives

$$\frac{K_{Ij} I}{K_{II} I} > C \frac{K_{I0} I}{K_{II} I} \quad \text{or} \quad \frac{K_{Ij}}{K_{I0}} > C$$

Both of the following parts require the calculation of K_{I0} which requires propagation of the membrane potential through the branchpoint. At the branchpoint, the input admittances of the 1 and 2 cylinders are

$$Y_1 = G_{\infty 1} q \tanh qL \quad \text{and} \quad Y_2 = G_{\infty 2} q \tanh qL \ ,$$

where *L* is the common electrotonic length of all the cylinders and the equations are simplified because of the closed-end boundary conditions. The transfer impedance K_{I0} is given by $K_{IB}A_{B0}$, where *B* is the branchpoint and A_{B0} is the voltage gain from *B* to 0. Using formulas discussed in class

$$K_{IB} = \frac{1}{\left(G_{\infty 1} q \tanh qL + G_{\infty 2} q \tanh qL\right)\cosh qL + G_{\infty 3} q \sinh qL} = \frac{1}{\left(G_{\infty 1} q + G_{\infty 2} q + G_{\infty 3} q\right)\sinh qL}$$

$$A_{B0} = \frac{1}{\cosh qL}$$

so

$$K_{I0} = K_{IB} A_{B0} = \frac{1}{\left(G_{\infty 1} q + G_{\infty 2} q + G_{\infty 3} q\right)\sinh qL \cosh qL} \quad . \tag{1}$$

Part b) The equation to be solved is $K_{IX}/K_{I0} > C$, where K_{IX} is the transfer impedance across the distance L_X in cylinder 3.

$$K_{IX} = \frac{1}{Y_X \cosh qL_X + G_{\infty 3} q \sinh qL_X} \quad , \tag{2}$$

where Y_X is the input admittance at the end of the subunit (L_X from the end of cylinder 3). It is llikely that at D.C. ($q=1$), Eqn. (2) is a monotonic decreasing function of L_X. Thus the equation $K_{IX}/K_{I0} > C$ will have a unique solution where $K_{IX} = CK_{I0}$. Finding Y_X is a mess, because it is the input admittance of the cylinder of length (L-L_X) between the end of the subunit and the branchpoint, and is given by

$$Y_X = G_{\infty 3} q \frac{\dfrac{(G_{\infty 1} q \tanh qL + G_{\infty 2} q \tanh qL)}{G_{\infty 3} q} + \tanh q(L - L_X)}{1 + \dfrac{(G_{\infty 1} q \tanh qL + G_{\infty 2} q \tanh qL)}{G_{\infty 3} q} \tanh q(L - L_X)} \quad . \tag{3}$$

So the condition to be solved for L_X is assembled from a combination of Eqns. (1), (2), and (3).

Part c) In this case, there are two conditions to be solved

$$K_{IY}/K_{I0} > C \quad \text{and} \quad K_{IZ}/K_{I0} > C$$

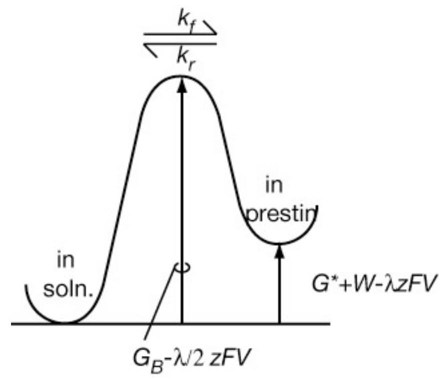
K_{I0} is given by Eqn. (1). K_{IY} and K_{IZ} take the same form for this problem, so only the first will be written out below.

$$K_{IY} = K_{IB}A_{BY} = K_{IB}\frac{1}{\cosh qL_Y + \dfrac{G_{\infty 1}q\tanh(L-L_Y)}{G_{\infty 1}q}\sinh qL_Y}$$
$$= K_{IB}\frac{1}{\cosh qL_Y + \tanh(L-L_Y)\sinh qL_Y} \qquad (4)$$

and K_{IB} is given by the equation derived in part a). The equation for K_{IZ} results after replacing L_Y in Eqn. (4) by L_Z, $G_{\infty 1}$ by $G_{\infty 3}$, and $(L\text{-}L_Y)$ by $(L\text{-}L_Z)$.

Problem 4

Part a) The barrier diagram is at right. The membrane potential terms in the energy diagram are negative because the Cl⁻ moves in the outward ($-V$) direction when it moves into prestin. From the diagram the parameters requested in the problem statement can be written:



$$G_S = G^* + W - \lambda zFV$$
$$k_f = \alpha e^{-(G_B-\lambda/2\,zFV)/RT} = k_{f0}e^{\lambda/2\,zFV/RT}$$
$$k_r = \alpha e^{-(G_B-\lambda/2\,zFV-G^*-W+\lambda zFV)/RT} = \alpha e^{-(G_B-G^*-W+\lambda/2\,zFV)/RT} = k_{r0}e^{-\lambda/2\,zFV/RT},$$

where k_{f0} and k_{r0} contain all the constant parameters. To simplify further, let $\beta = \lambda F/2RT$, then, using the fact that z=-1

$$k_f = k_{f0}e^{-\beta V} \quad \text{and} \quad k_r = k_{r0}e^{\beta V}$$
$$k_{f0} = \alpha e^{-G_B/RT} \quad \text{and} \quad k_{r0} = \alpha e^{-(G_B-G^*-W)/RT} .$$

The usual differential equation is

$$\frac{d\Pi}{dt} = k_f[Cl]\mathrm{X} - k_r\Pi .$$

To avoid confusion, X is the fraction of prestins with no Cl⁻ and Π is the fraction of prestins with Cl⁻. As usual, X + Π = 1, so that

$$\frac{d\Pi}{dt} = k_f' - (k_f' + k_r)\Pi \ .$$

Where $k_f' = k_f\,Cl$; because the chloride concentration is a constant, it can be absorbed into k_f.

Part b) In the steady state with $V{=}V_0$ and $\Pi{=}\Pi_0$

$$\frac{d\Pi}{dt} = k_f' - (k_f' + k_r)\Pi = 0 \quad \text{so that} \quad k_f'(V_0) - \left(k_f'(V_0) + k_r(V_0)\right)\Pi_0 = 0 \ ,$$

where the dependence of the rate constants on V is noted.

Part c) Substituting the voltage dependence gives the steady-state relationship

$$k_{f0}' e^{-\beta V_0} - \left[k_{f0}' e^{-\beta V_0} + k_{r0} e^{\beta V_0}\right]\Pi_0 = 0 \quad . \tag{*}$$

Differentiating this equation w.r.t V and Π gives the change in Π that occurs in response to a change in V to keep the system in steady state,

$$-k_{f0}' e^{-\beta V_0}\beta v - \left[-k_{f0}' e^{-\beta V_0}\beta v + k_{r0} e^{\beta V_0}\beta v\right]\Pi_0 - \left[k_{f0}' e^{-\beta V_0} + k_{r0} e^{\beta V_0}\right]\pi = 0 \ .$$

The differentiation uses the notation $dV{=}v$ and $d\Pi{=}\pi$. The equation can be simplified again using the steady-state relationship. Add

$$k_{r0} e^{\beta V_0}\beta v - k_{r0} e^{\beta V_0}\beta v$$

to the first term in brackets giving

$$-k_{f0}' e^{-\beta V_0}\beta v + \left[k_{f0}' e^{-\beta V_0}\beta v + k_{r0} e^{\beta V_0}\beta v\right]\Pi_0 - 2k_{r0} e^{\beta V_0}\beta v\,\Pi_0 - \left[k_{f0}' e^{-\beta V_0} + k_{r0} e^{\beta V_0}\right]\pi = 0 \ .$$

The first two terms are again zero from the steady-state relationship, so

$$-2k_{r0} e^{\beta V_0}\beta v\,\Pi_0 - \left[k_{f0}' e^{-\beta V_0} + k_{r0} e^{\beta V_0}\right]\pi = 0 \,.$$

This is a relationship between the small-signal variables that can be recast as

$$-\pi = \frac{2k_{r0} e^{\beta V_0}\beta\Pi_0}{\left[k_{f0}' e^{-\beta V_0} + k_{r0} e^{\beta V_0}\right]} v \tag{**}$$

Part d) $-d\pi/dt$ is the small-signal rate of transfer of Cl^- into the membrane (moles/s/cm^2) and is therefore a current if multiplied by zF. Because chloride is negatively charged, the actual charge movement is $-Fd\pi/dt$. Differentiating Eqn. (*) in the exam problem, the same as Eqn. (**) above, gives

$$-zF\frac{d\pi}{dt} = I_{CapNL} = C_{NL}(V_0)\frac{dv}{dt} \ ,$$

which is the current-voltage relationship of a capacitor

$$C_{NL} = \frac{2F\,k_{r0}e^{\beta V_0}\beta\,\Pi_0}{\left[k'_{f0}e^{-\beta V_0} + k_{r0}e^{\beta V_0}\right]} \qquad (***)$$

Part e) At large negative voltage, all the prestin is saturated with Cl^- so there is no charge movement for small *v*; similarly, at large positive voltage, there is no Cl^- in the membrane and again there is no charge movement for small *v*.

To see the effect of membrane potential on the capacitance, note that the steady-state relationship (Eqn. (*)) allows P_0 in Eqn. (***) to be written as follows

$$C_{NL} = \frac{2F\,k_{r0}e^{\beta V_0}\beta}{\left[k'_{f0}e^{-\beta V_0} + k_{r0}e^{\beta V_0}\right]}\Pi_0 = \frac{2F\,k_{r0}e^{\beta V_0}\beta}{\left[k'_{f0}e^{-\beta V_0} + k_{r0}e^{\beta V_0}\right]}\frac{k'_{f0}e^{-\beta V_0}}{\left[k'_{f0}e^{-\beta V_0} + k_{r0}e^{\beta V_0}\right]}$$
$$= \frac{2F\,k'_{f0}k_{r0}\beta}{\left[k'_{f0}e^{-\beta V_0} + k_{r0}e^{\beta V_0}\right]^2}.$$

As $V_0 \rightarrow \infty$ or $V_0 \rightarrow -\infty$, $C_{NL} \rightarrow 0$ because of the exponentials in the denominator. This corresponds to the qualitative explanation given in the previous paragraph. The function looks like the figure in the problem set, if graphed, with a peak capacitance at $V_0 = RT/F\,\ln(k'_{fo}/k_{ro})/\lambda$.