Chapter 12 Review Questions and Problems

9–12 HEAT PROBLEM

Find the temperature \( w(x, t) \) in a semi-infinite laterally insulated bar extending from along the x-axis to infinity, assuming that the initial temperature is 0, \( w(x, t) \rightarrow 0 \) as \( x \rightarrow \infty \) for every fixed \( t \geq 0 \), and \( w(0, t) = f(t) \). Proceed as follows.

9. Set up the model and show that the Laplace transform leads to

\[
\mathcal{L}[w(x, t)] = \frac{c^2}{\alpha^2} \frac{\partial^2 W}{\partial x^2} \quad (W = \mathcal{L}[w])
\]

and

\[
W = F(s)e^{-\sqrt{c}x/c} \quad (F = \mathcal{L}[f]).
\]

10. Applying the convolution theorem, show that in Prob. 9,

\[
w(x, t) = \frac{x}{2c\sqrt{\pi}} \int_0^t f(t - \tau) \tau^{-3/2} e^{-x^2/4\tau}\,d\tau.
\]

11. Let \( w(0, t) = f(t) \) (Sec. 6.3). Denote the corresponding \( w, W, \) and \( F \) by \( w_0, W_0, \) and \( F_0 \). Show that then in Prob. 10,

\[
w_0(x, t) = \frac{x}{2c\sqrt{\pi}} \int_0^t \tau^{-3/2} e^{-x^2/4\tau}\,d\tau
\]

with the error function \( \text{erf} \) as defined in Problem Set 12.7.

12. Duhamel’s formula.\(^4\) Show that in Prob. 11,

\[
W_0(x, s) = \frac{1}{x} e^{-\sqrt{c}x/c}
\]

and the convolution theorem gives Duhamel’s formula

\[
w(x, t) = \int_0^t f(t - \tau) \frac{\partial w_0}{\partial \tau}\,d\tau.
\]

CHAPTER 12 REVIEW QUESTIONS AND PROBLEMS

1. For what kinds of problems will modeling lead to an ODE? To a PDE?

2. Mention some of the basic physical principles or laws that will give a PDE in modeling.

3. State three or four of the most important PDEs and their main applications.

4. What is “separating variables” in a PDE? When did we apply it twice in succession?

5. What is d’Alembert’s solution method? To what PDE does it apply?

6. What role did Fourier series play in this chapter? Fourier integrals?

7. When and why did Legendre’s equation occur? Bessel’s equation?

8. What are the eigenfunctions and their frequencies of the vibrating string? Of the vibrating membrane?

9. What do you remember about types of PDEs? Normal forms? Why is this important?

10. When did we use polar coordinates? Cylindrical coordinates?

11. Explain mathematically (not physically) why we got exponential functions in separating the heat equation, but not for the wave equation.

12. Why and where did the error function occur?

13. How do problems for the wave equation and the heat equation differ regarding additional conditions?

14. Name and explain the three kinds of boundary conditions for Laplace’s equation.

15. Explain how the Laplace transform applies to PDEs.

16–18 Solve for \( u = u(x, y) \):

16. \( u_{xx} + 25u = 0 \)

17. \( u_{yy} + u_y - 6u = 18 \)

18. \( u_{xx} + u_x = 0, \, u(0, y) = f(y), \, u_x(0, y) = g(y) \)

19–21 NORMAL FORM

Transform to normal form and solve:

19. \( u_{xy} = u_{yy} \)

20. \( u_{xx} + 6u_{xy} + 9u_{yy} = 0 \)

21. \( u_{xx} - 4u_{yy} = 0 \)

22–24 VIBRATING STRING

Find and sketch or graph (as in Fig. 288 in Sec. 12.3) the deflection \( u(x, t) \) of a vibrating string of length \( \pi \), extending from \( x = 0 \) to \( x = \pi \), and \( c^2 = T/\rho = 4 \) starting with velocity zero and deflection:

22. \( \sin 4x \)

23. \( \sin^3 x \)

24. \( \frac{1}{2} \pi - |x - \frac{1}{2} \pi| \)

\(^4\)JEAN–MARIE CONSTANT DUHAMEL (1797–1872), French mathematician.
25–27 HEAT
Find the temperature distribution in a laterally insulated thin copper bar (\(c^2 = k/(\pi \rho) = 1.158 \text{ cm}^2/\text{sec}\)) of length 100 cm and constant cross section with endpoints at \(x = 0\) and 100 kept at 0°C and initial temperature:
25. \(\sin 0.01 \pi x\)
26. \(50 - |50 - x|\)
27. \(\sin^3 0.01 \pi x\)

28–30 ADIABATIC CONDITIONS
Find the temperature distribution in a laterally insulated bar of length \(\pi\) with \(c^2 = 1\) for the adiabatic boundary condition (see Problem Set 12.6) and initial temperature:
28. \(3x^2\)
29. \(100 \cos 2x\)
30. \(2\pi - 4|x - \frac{1}{2} \pi|\)

31–32 TEMPERATURE IN A PLATE
31. Let \(f(x, y) = u(x, y, 0)\) be the initial temperature in a thin square plate of side \(\pi\) with edges kept at 0°C and faces perfectly insulated. Separating variables, obtain from \(u_t = c^2 \nabla^2 u\) the solution
\[
u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \sin mx \sin ny e^{-c^2 (m^2 + n^2) t}
\]
where
\[
B_{mn} = \frac{4}{\pi^2} \int_0^\pi \int_0^\pi f(x, y) \sin mx \sin ny \, dx \, dy.
\]
32. Find the temperature in Prob. 31 if \(f(x, y) = x(\pi - x)y(\pi - y)\).

33–37 MEMBRANES
Show that the following membranes of area 1 with \(c^2 = 1\) have the frequencies of the fundamental mode as given (4-decimal values). Compare.
33. Circle: \(\alpha_1/(2 \sqrt{\pi}) = 0.6784\)
34. Square: \(1/\sqrt{2} = 0.7071\)
35. Rectangle with sides 1:2:3/\(\sqrt{8} = 0.7906\)
36. Semicircle: \(3.832/\sqrt{8 \pi} = 0.7643\)
37. Quadrant of circle: \(\alpha_{21}/(4 \sqrt{\pi}) = 0.7244\)
\(\alpha_{21} = 5.13562\) = first positive zero of \(J_2\)

38–40 ELECTROSTATIC POTENTIAL
Find the potential in the following charge-free regions.
38. Between two concentric spheres of radii \(r_0\) and \(r_1\) kept at potentials \(u_0\) and \(u_1\), respectively.
39. Between two coaxial circular cylinders of radii \(r_0\) and \(r_1\) kept at the potentials \(u_0\) and \(u_1\), respectively. Compare with Prob. 38.
40. In the interior of a sphere of radius 1 kept at the potential \(f(\phi) = \cos 3\phi + 3 \cos \phi\) (referred to our usual spherical coordinates).

SUMMARY OF CHAPTER 12

Partial Differential Equations (PDEs)

Whereas ODEs (Chaps. 1–6) serve as models of problems involving only one independent variable, problems involving two or more independent variables (space variables or time \(t\) and one or several space variables) lead to PDEs. This accounts for the enormous importance of PDEs to the engineer and physicist. Most important are:

1. \(u_{tt} = c^2 u_{xx}\) One-dimensional wave equation (Secs. 12.2–12.4)
2. \(u_{tt} = c^2 (u_{xx} + u_{yy})\) Two-dimensional wave equation (Secs. 12.8–12.10)
3. \(u_t = c^2 u_{xx}\) One-dimensional heat equation (Secs. 12.5, 12.6, 12.7)
4. \(\nabla^2 u = u_{xx} + u_{yy} = 0\) Two-dimensional Laplace equation (Secs. 12.6, 12.10)
5. \(\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = 0\) Three-dimensional Laplace equation (Sec. 12.11).

Equations (1) and (2) are hyperbolic, (3) is parabolic, (4) and (5) are elliptic.
In practice, one is interested in obtaining the solution of such an equation in a given region satisfying given additional conditions, such as initial conditions (conditions at time \( t = 0 \)) or boundary conditions (prescribed values of the solution \( u \) or some of its derivatives on the boundary surface \( S \), or boundary curve \( C \), of the region) or both. For (1) and (2) one prescribes two initial conditions (initial displacement and initial velocity). For (3) one prescribes the initial temperature distribution. For (4) and (5) one prescribes a boundary condition and calls the resulting problem a (see Sec. 12.6)

- **Dirichlet problem** if \( u \) is prescribed on \( S \),
- **Neumann problem** if \( u_n = \partial u / \partial n \) is prescribed on \( S \),
- **Mixed problem** if \( u \) is prescribed on one part of \( S \) and \( u_n \) on the other.

A general method for solving such problems is the method of separating variables or product method, in which one assumes solutions in the form of products of functions each depending on one variable only. Thus equation (1) is solved by setting \( u(x, t) = F(x)G(t) \); see Sec. 12.3; similarly for (3) (see Sec. 12.6). Substitution into the given equation yields ordinary differential equations for \( F \) and \( G \), and from these one gets infinitely many solutions \( F = F_n \) and \( G = G_n \) such that the corresponding functions

\[
  u_n(x, t) = F_n(x)G_n(t)
\]

are solutions of the PDE satisfying the given boundary conditions. These are the eigenfunctions of the problem, and the corresponding eigenvalues determine the frequency of the vibration (or the rapidity of the decrease of temperature in the case of the heat equation, etc.). To satisfy also the initial condition (or conditions), one must consider infinite series of the \( u_n \), whose coefficients turn out to be the Fourier coefficients of the functions \( f \) and \( g \) representing the given initial conditions (Secs. 12.3, 12.6). Hence Fourier series (and Fourier integrals) are of basic importance here (Secs. 12.3, 12.6, 12.7, 12.9).

**Steady-state problems** are problems in which the solution does not depend on time \( t \). For these, the heat equation \( u_t = c^2 \nabla^2 u \) becomes the Laplace equation.

Before solving an initial or boundary value problem, one often transforms the PDE into coordinates in which the boundary of the region considered is given by simple formulas. Thus in polar coordinates given by \( x = r \cos \theta \), \( y = r \sin \theta \), the Laplacian becomes (Sec. 12.11)

\[
  \nabla^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta \theta}.
\]

for spherical coordinates see Sec. 12.10. If one now separates the variables, one gets Bessel’s equation from (2) and (6) (vibrating circular membrane, Sec. 12.10) and Legendre’s equation from (5) transformed into spherical coordinates (Sec. 12.11).