The Johns Hopkins University
Department of Electrical and Computer Engineering

505.460 — Introduction to Linear Systems — Spring 1996

Quiz No. 2 – Answers.

1. *Compute the complex Fourier coefficients C_k for the signal $x(t)$ given below [Picture deleted].* **Answer:** First of all, the signal has period $T_0 = 3$. Secondly, the signal, over the period $0 \le t \le 3$ can be defined as $x(t) = \delta(t-1) + u(t-2) - u(t-3)$. The Fourier series coefficients are:

$$\begin{aligned} C_k &= \frac{1}{3}\int_0^3 [\delta(t-1) + u(t-2) - u(t-3)]e^{-j2\pi kt/3}\,dt \\ &= \frac{1}{3}e^{-j2\pi k/3} + \frac{1}{3}\frac{e^{-j2\pi kt/3}}{-j2\pi k/3}\bigg|_2^3 \\ &= \frac{1}{3}e^{-j2\pi k/3} + \frac{j}{2\pi k}(e^{-j2\pi k} - e^{-j4\pi k}) \\ &= \frac{1}{3}e^{-j2\pi k/3} + \frac{j}{2\pi k}(1 - e^{-j4\pi k}) \end{aligned}$$

2. *Compute the Fourier transform of the signal $x(t) = 3\delta(t-1) + e^{-2(t-1)}u(t-1)$.*
Answer: Note that $x(t) = z(t-1)$ where $z(t) = 3z_1(t) + z_2(t)$, $z_1(t) = \delta(t)$, and $z_2(t) = e^{-2t}u(t)$. Using tables or the definition of the FT, it is easy to check that $\mathcal{F}[z_1] = 1$ and $\mathcal{F}[z_2] = 1/(2+j\omega)$. Also, $\mathcal{F}[x] = e^{-j\omega}\mathcal{F}[z]$ and so:

$$\mathcal{F}[x(t)] = e^{-j\omega}\frac{7+3j\omega}{2+j\omega}$$

3. *If $x(t) = \sin^2(3\pi t)$. (a) What is the fundamendal period T_0?* **Answer:** Using Euler's relation: $\sin^2(3\pi t) = \frac{1}{2} - \frac{1}{2}\cos(6\pi t)$ and so the signal has period $T_0 = 2\pi/w_0 = 2\pi/(6\pi) = 1/3$. *(b) Compute the coefficients of the Fourier Series.* **Answer:** Using the relation above, we clearly have:

$$x(t) = \frac{1}{2} + \frac{1}{4}(e^{j6\pi t} + e^{-j6\pi t})$$

This *is* the FS of $x(t)$ and so $C_0 = 1/2$ and $C_2 = C_{-2} = 1/4$. +

4. *Suppose that the input to a continuous-time, linear time-invariant system is given by: $x(t) = 2e^{-3t}u(t)$. The output equals $y(t) = e^{-t}u(t) - e^{-3t}u(t)$. Find the impulse response of the system $h(t)$.*
Answer: Using the tables, $\mathcal{F}[e^{-at}u(t)] = 1/(a+j\omega)$. Thus, $Y(\omega) = 1/(1+j\omega) - 1/(3+j\omega) = 2/[(1+j\omega)(3+j\omega)]$ and $X(\omega) = 2/(3+j\omega)$. The FT of the impulse response of the system is then

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{2}{(1+j\omega)(3+j\omega)}\frac{3+j\omega}{2} = \frac{1}{1+j\omega}$$

Taking the inverse FT of $H(\omega)$ we get $h(t) = e^{-t}u(t)$.

Chapter 6 – Answers.

6.20 Signal $Y_1(\omega)$ consists of $[X(w-\omega_c)+X(\omega+\omega_c)]/2$. Signal $H(\omega)X(\omega)$ consists of $-jX(\omega)$ for $\omega \geq 0$ and $jX(\omega)$ for $\omega \leq 0$. Signal $Y_2(\omega)$ can be written as:

$$Y_2(\omega) = \begin{cases} \frac{-j}{2j}X(\omega-\omega_c) & \omega \in (\omega_c, \omega_c+\omega_m) \\ \frac{j}{2j}X(\omega-\omega_c) & \omega \in (\omega_c-\omega_m, \omega_m) \\ \frac{-j}{-2j}X(\omega+\omega_c) & \omega \in (-\omega_c, -\omega_c-\omega_m) \\ \frac{j}{-2j}X(\omega+\omega_c) & \omega \in (-\omega_c-\omega_m, -\omega_c) \end{cases}$$

Adding the two signals results in:

$$Y(\omega) = \begin{cases} (\frac{-1}{2}+\frac{1}{2})X(\omega-\omega_c) = 0 & \omega \in (\omega_c, \omega_c+\omega_m) \\ (\frac{1}{2}+\frac{1}{2})X(\omega-\omega_c) = X(\omega-\omega_c) & \omega \in (\omega_c-\omega_m, \omega_m) \\ (\frac{1}{2}+\frac{1}{2})X(\omega+\omega_c) = X(\omega+\omega_c) & \omega \in (-\omega_c, -\omega_c-\omega_m) \\ (\frac{-1}{2}+\frac{1}{2})X(\omega+\omega_c) = 0 & \omega \in (-\omega_c-\omega_m, -\omega_c) \end{cases}$$

That is, we are left with the lower side-bands.

6.26 (a) If $x(t)$ is band-limited by bandwidth W (i.e. the function $X(\omega)=0$ for $|\omega|>W$, then the sampling frequence $\omega_s = 2\pi/T$ must be at least $2W$. Thus $T < \pi/W$. (b) $H(\omega) = T$ for $|\omega| \leq W$ and zero elsewhere. (c) If $x(t) = \cos[2\pi(50)t]$, then $W/[2\pi] = 50$ Hz. The sampling frequency is $f_s = \omega_s/[2\pi] = 1/T = 1/0.004 = 250$ Hz. The frequencies of x_p less than 725 Hz are caused by two components. The part of the sinusoid at 50Hz that is shifted by multiples of 250Hz, and the part of the sinusoid at -50Hz that is also shifted by multiples of 250Hz. These are 50, 300, 550 Hz and 200, 450 and 700 Hz respectively. (d) If $\cos(2\pi f_x t)$ is such that $f_x = 300Hz$, the frequency components of x_p are those of 300 Hz shifted by multiples of 250 Hz: 50,300,550 and those of -300Hz shifted by multiples of 250: -50,200,450,700. Clearly this set is the same as that of (c).

6.27 Note that $M(\omega) = 2\delta(\omega-200\pi)+2\delta(\omega+200\pi)+4/j\delta(\omega-800\pi)-4/j\delta(\omega+800\pi)$. If we sample at 2000π rad/s. We are going to get copies of each of these 4 subsignals, spaced 2000π rad/s apart, scaled by $1/T$ where $T=0.001$. The components will be at frequencies:

$$\{\cdots, -3800, -1800, 200, 2200, 3200\cdots\} \cup \{\cdots, -4200, -2200, -200, 1800, 3800\cdots\}$$
$$\cup\{\cdots, -3200, -1200, 800, 2800, 4800\cdots\} \cup \{\cdots, -4800, -2800, -800, 1200, 3200\cdots\}$$

(b) As above, the components will be at frequencies:

$$\{\cdots, -1800, -800, 200, 1200, 2200\cdots\} \cup \{\cdots, -2200, -1200, -200, 800, 1800\cdots\}$$
$$\cup\{\cdots, -1200, -200, 800, 1800, 2800\cdots\} \cup \{\cdots, -2800, -1800, -800, 200, 1200\cdots\}$$

(c) The highest frequeny component is the 800π rad/s (400Hz). You need to sample at least twice this frequency: 1600π rad/sec or 800 Hz.