Formal Methods: Set Theory

2.1 Naive Set Theory

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The Naive Concept of a Set

(Pseudo) Def 1.1.1. A **set** is a collection of elements. Examples:

- $\{5, 7, \text{Plato}\}$
- $\{x \mid x \text{ is a Canadian citizen}\}$, or, alternatively,
- $\{x : x \text{ is a Canadian citizen}\}$
- $\{\{5, 7\}, \{\text{Plato}\}, \text{Plato}\}$
- $\mathbb{N} = \{n \mid n \text{ is a natural number}\} = \{0, 1, 2, \ldots\}$

N.B. The specification of a set by listing its members can always be done away with by specifying a property. E.g.,

$$\{5, 7, \text{Plato}\} = \{x \mid x = 5 \text{ or } x = 7 \text{ or } x = \text{Plato}\}.$$

Perhaps Worrisome Questions

Just what is a *collection*?

For that matter, what is an *element*?

E.g., could the entire universe, if there is such a thing, be an element?

For the time being set these worries aside.

Some Basic Concepts and Notation

(Pseudo) Def 1.1.2. Let A be a set. Then A is a collection of elements. If b is one of these, then we say that b is a **member** of A and write:

$$b \in A.$$

Example: $\text{Plato} \in \{\text{Plato}, \text{Aristotle}\}$.

Def 1.1.3. If A and B are sets and every member of A is a member of B, then we say that A is a **subset** of B and write:

$$A \subseteq B.$$

If $A \subseteq B$ but $A \neq B$, A is a **proper subset** of B. Notation: $A \subset B$.

Example: $\{\text{Aristotle}\} \subset \{\text{Plato}, \text{Aristotle}\}$.

Def 1.1.4. The **empty set** is $\emptyset =_{df} \{x \mid x \neq x\}$, i.e., the set with *no* members.

N.B. For any set A, both $A \subseteq A$ and $\emptyset \subseteq A$.

Some Set Operations

Def 1.1.5. The **union** of A and B is

$$A \cup B =_{df} \{x \mid x \in A \text{ or } x \in B\}.$$

Example: $\{\text{Plato}\} \cup \{\text{Aristotle}\} = \{\text{Plato}, \text{Aristotle}\}$.

Def 1.1.6. The **intersection** of A and B is

$$A \cap B =_{df} \{x \mid x \in A \text{ and } x \in B\}.$$

Example: $\{\text{Plate}\} \cap \{\text{Aristotle}\} = \emptyset$, i.e., $\{\text{Plato}\}$ and $\{\text{Aristotle}\}$ are **disjoint**.

Def 1.1.7. The **relative complement** (**set difference**) of A with B is

$$A \setminus B =_{df} \{x \mid x \in A \text{ and } x \notin B\}.$$

Example: $\{\text{Plato}, \text{Aristotle}\} \setminus \{\text{Plato}\} = \{\text{Aristotle}\}$.

Some More Set Operations

Def 1.1.8. The **power set** of A is

$$\mathcal{P}(A) = \{X \mid X \subseteq A\}.$$

Example: $\mathcal{P}(\{0,1,2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$.

N.B. Some people like to write 2^A for $\mathcal{P}(A)$. We'll see the rationale for this notation in due time.

Def 1.1.9. Let $\mathcal{F}$ be a family of sets. The **generalized union** of $\mathcal{F}$ is

$$\bigcup \mathcal{F} =_{df} \{x \mid x \in Y \text{ for some } Y \in \mathcal{F}\}.$$

Def 1.1.10. Assume $\mathcal{F} \neq \emptyset$. The **generalized intersection** of $\mathcal{F}$ is

$$\bigcap \mathcal{F} =_{df} \{x \mid x \in Y \text{ for all } Y \in \mathcal{F}\}.$$

Explanation of the Generalized Operations

If $\mathcal{F}$ is finite, say $\mathcal{F} = \{A_1, \ldots, A_n\}$, then

$$\bigcup \mathcal{F} = A_1 \cup \cdots \cup A_n$$

and

$$\bigcap \mathcal{F} = A_1 \cap \cdots \cap A_n.$$

If $\mathcal{F}$ is infinite and can be enumerated $\mathcal{F} = \{A_1, A_2, \ldots\}$, then

$$\bigcup \mathcal{F} = A_1 \cup A_2 \cup \cdots$$

and

$$\bigcap \mathcal{F} = A_1 \cap A_2 \cap \cdots .$$

N.B. It may be that $\mathcal{F}$ cannot be enumerated. More on this later.

Critique of Naive Set Theory

- The idea starting with a definition of 'set' follows the model of Euclid's geometry: a *point* is that which has no parts.
- But starting this way presupposes that the terms in the *definiens* are well-defined. How can this be?
- If they themselves have definitions, then the terms in the *definiens* of each of them must have definitions, and so on, *ad infinitum*. This is a vicious regress.
- Therefore the terms in the initial *definiens* must be understandable on their own terms.
- We defined a set as a *collection* of *elements*. But 'collection' is no more illuminating than 'set' and 'element' no more illuminating than 'member'.

Critique of Naive Set Theory (cont.)

- So we must already know what a set is.
- But the notion of a set is an abstract and not an empirical concept.
- Where does this abstract knowledge come from?
- Even if we think we know what a set is, or let us say that we know of particular sets, say X and Y, that they *are* sets, what guarantees that $X \cup Y$, $X \cap Y$, $\bigcup X$ and $\mathcal{P}(X)$ are sets that exist, or for that matter even $\{X, Y\}$?
- How do we know there is an infinite set such as $\mathbb{N}$?

Refutation of Naive Set Theory

- We have been assuming that for any property P there is a set containing all and only the things that have property P.
- Call this the **principle of abstraction**.
- This principle was implicitly behind the use of the notation

$$\{x \mid P(x)\},$$

  where '$P(x)$' means that x has the property P.
- This assumption leads to disastrous consequences, namely that Naive Set Theory is INCONSISTENT.

Russell's Paradox

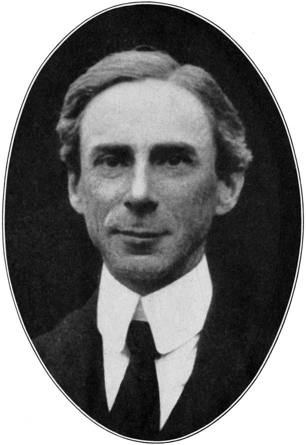

Figure: Bertrand Russell (1872–1970)

Russell's Paradox (cont.)

- Let $P(x)$ be the property of *not* being a member of one's self, i.e., x has the property P iff $x \notin x$.
- Then we have the set

$$R =_{df} \{x \mid x \notin x\}.$$

- Now ask whether or not $R \in R$.
- If $R \in R$, then, because any $x \in R$ is such that $x \notin x$, it follows that $R \notin R$.
- But if $R \notin R$, then it has the property of *not* being a member of itself, and, since R has for its members all and only those things that have this property, it follows that $R \in R$.
- Therefore, $R \in R$ if and only if $R \notin R$, which is classically inconsistent.