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CATEGORICAL ALGEBRA AND SET-THEORETIC FOUNDATIONS

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1. Introduction. This note will speculate on some of the ways in which the current practice of category theory may suggest or require revisions in the use of axiomatic set theory as a foundation for mathematical practice.

Categorical algebra has developed in recent years as an effective method of organizing parts of mathematics. Typically, this sort of organization uses notions such as that of the category **G** of all groups. This category consists of two collections: The collection of all groups G and the collection of all homomorphisms $t: G \to H$ of one group G into another one; the basic operation in this category is the composition of two such homomorphisms. To realize the intent of this construction it is vital that this collection **G** contain *all* groups; however, if "collection" is to mean "set" in any one of the usual axiomatizations of set-theory, this intent cannot be directly realized. This raises the problem of finding some axiomatization of set theory—or of some foundational discipline like set theory—which will be adequate and appropriate to realizing this intent. This problem may turn out to have revolutionary implications vis-a-vis the accepted views of the role of set theory.

2. Categories via axioms. The definition and elementary properties of a category involve no real foundational problems. This may be indicated by reformulating in axiomatic terms, free of set theory, one of the standard descriptions of a category. (An introductory exposition of categories is given in an algebra text by Mac Lane-Birkhoff **[15]**; the more extensive standard expositions are

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those by Freyd [**2**], Mac Lane [**14**], and Mitchell [**16**].) A *category* **C** is a two-sorted system, the sorts being called *objects* A of **C** and *morphisms* f of **C**. The undefined terms, in context, are "A is the *domain* (or, the *codomain*) of f", "k is the *composite* of g with f", and "ι is the *identity* morphism of A". The first axiom asserts that every morphism f has exactly one object A as domain and one object B as codomain; when this is the case, we write $f: A \to B$. Next, for all morphisms g and f, there exists a composite k of g with f if and only if domain $(g) =$ codomain (f); when this is the case, the composite k is unique, and is written as $k = gf$. Another axiom requires that this composition be associative $[h(gf) = (hg)f]$ whenever possible. Every object B has one and only one identity morphism, with domain and codomain B, which is written as $1_B: B \to B$. Finally, $1_B f = f$ for every $f: A \to B$ and $g 1_B = g$ for every $g: B \to C$.

A prime example is "the" category **S** of sets (more exactly, the category defined from any (model of *a*) suitable set theory). In this category the objects are the sets S and the morphisms are the functions $f: S \to T$, with the usual domain, identity functions and composition. Note explicitly that the notion of function is *not* that customary in axiomatic set theory; indeed, for categorical purposes a morphism (function) has *both* a given domain *and* a given codomain. This means that a function must be described as a suitable set of ordered pairs *plus* a suitable set as codomain. For example, let S be a proper subset of T and $D \subset S \times S$ the set of all ordered pairs (s, s) for $s \in S$. Then (S, D, S) is (the graph of) the usual identity function $1_S: S \to S$, while (S, D, T) is (the graph of) the injection ("insertion") $i_{S,T}: S \to T$ (that function which maps S as a subset into T). For categorical purposes 1_S and $i_{S,T}$ must be treated as *different* functions, essentially because they behave differently under functors (for a detailed explanation, see Mac Lane-Birkhoff [**15**, p. 325]).

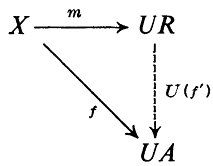
A morphism of categories is called a *functor*. In detail, a functor $F: \mathbf{C} \to \mathbf{D}$ assigns to each object A of **C** an object $D = F(A)$ of **D** and to each morphism $:A \to B$ of **C** a morphism $d = F(f): F(A) \to F(B)$ of **D**. Here "assigns", viewed axiomatically, is a locution for the following undefined terms, "D is the image of A under F" and "d is the image of f under F". The axioms for a functor F require that $F(1_B) = 1_{FB}$ for every B and that $F(gf) = F(g)F(f)$ whenever the composite gf is defined.

A morphism of functors is called a *natural transformation*. In detail, if **C** and **D** are categories and $F: \mathbf{C} \to \mathbf{D}$ and $F': \mathbf{C} \to \mathbf{D}$ are functors, then a natural transformation $\theta: F \to F'$ "assigns" to each object A of **C** a morphism $\theta_A: F(A) \to F'(A)$ in **D** in such fashion that the equality $\theta_B F(f) = F'(f)\theta_A$ (of composites in **D**) holds for every morphism $f: A \to B$ of **C**. This equality is usually pictured by the statement that the following square diagram commutes:

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ \downarrow{\scriptstyle \theta_A} & & \downarrow{\scriptstyle \theta_B} \\ F'(A) & \xrightarrow[F'(f)]{} & F'(B) \end{array}$$

This commutative diagram expresses the "naturality" of θ.

A final basic notion is that of adjoint functor, due to Kan [**8**]. It may be described in terms of the closely related notion of a "universal construction", due to Samuel [**18**] and Bourbaki. Given a function $U: \mathbf{C} \to \mathbf{X}$, and an object X of $\mathbf{X}$, a *universal morphism* from X under U is a morphism $m: X \to UR$, with R an object of $\mathbf{C}$, such that every morphism $f: X \to UA$ in $\mathbf{X}$ can be written as a composite $f = U(f')m$ for exactly one morphism $f': R \to A$ of $\mathbf{C}$. The corresponding commutative diagram is



A left adjoint of the given functor $U: \mathbf{C} \to \mathbf{X}$ is then described as a functor $F: \mathbf{X} \to \mathbf{C}$ (in the opposite direction) for which there is a natural transformation $\eta: I \to UF$, $I: \mathbf{X} \to \mathbf{X}$ the identity functor, and such that each $\eta_X: X \to UFX$ is universal from X under U. The multiplicity of working examples of adjoint functors is matched by the protean forms of their definitions, as given in the standard references cited above.

3. Categories via sets and classes. In their original introduction of the notion of a category [**1**], Eilenberg-Mac Lane noted that the category of all sets could not be described legitimately as a set, and so proposed to describe it as a class within a Gödel-Bernays set theory (in which *both* "set" and "class" are primitive notions, as in Rubin [**17**]). Call such a category "large". In detail, then, a *large* category is a class of objects and a class of morphisms, with domain, composition, etc. all such as to satisfy the axioms given above for a category. Take the set theory in the version where sets are regarded as special classes, and call a category $\mathbf{C}$ *small* when the class of all its morphisms is a set; since every object has an identity morphism, this also implies that the class of all objects of $\mathbf{C}$ is a set. More generally, for any pair of objects A and B in a large category the comprehension axiom for classes constructs the class

$$\hom_{\mathbf{C}}(A, B) = \{f \mid f \text{ a morphism of } \mathbf{C} \text{ and } f: A \to B\}.$$

A (large) category is said to be locally small (Mitchell's different and aberrant use of this term is to be rejected) when each class $\hom_{\mathbf{C}}(A, B)$ is a set. Usually, the definition of a category is given in terms of such "hom-sets", and so defines only these locally small categories. In particular, the category $\mathbf{S}$ of all sets (of our underlying Gödel-Bernays set theory) is a large category which is locally small (given sets S and T, $\hom(S, T) = T^S$ is a set).

The same small-large distinction may be drawn for other, more familiar, algebraic objects. Thus a "large group" is a *class* G equipped with a function $G \times G \to G$ which satisfies the usual axioms for multiplication in a group, while a "small group" is one for which this class G is actually a set. Now the comprehension axiom of Gödel-Bernays set theory allows one to form the large category of all small groups, in which the class of objects is the class of all (small) groups.

Again, this is a locally small category. There are many other useful examples of locally small categories with objects all (small) Mathematical systems of a fixed sort and morphisms all homomorphisms of these systems. A detailed indication of the construction of these categories by Gödel-Bernays set theory is given in Mac Lane-Birkhoff [**15**, Chapter XV].

Two such constructions of large categories are especially useful. One is **Cat**: The category of all small categories; with objects all small categories and morphisms all functors. For many purposes, as indicated below, one would like to have a bigger **cat**, say the category of all large categories or of all categories *überhaupt*. Another one is the *functor category* $\mathbf{A}^{\mathbf{B}}$ determined by a large category $\mathbf{A}$ and a small category $\mathbf{B}$. Its objects are all functors $F: \mathbf{B} \to \mathbf{A}$ and its morphisms $F \to F'$ are all natural transformations $\theta: F \to F'$. Since the domain category $\mathbf{B}$ is small, each functor $F: \mathbf{B} \to \mathbf{A}$ can be represented by a set, say the usual graph of F; hence one may form the class of all such F, making $\mathbf{A}^{\mathbf{B}}$ a locally small category. There are, however, many properties of large categories $\mathbf{A}$ and $\mathbf{B}$ which can be effectively visualized in the (superlarge (?)) category $\mathbf{A}^{\mathbf{B}}$.

4. Categories in universes. Grothendieck and his followers have made extensive use of categories in algebraic geometry and in this connection have proposed an alternative way of treating large categories within Zermelo-Fraenkel set theory. Define a *universe* to be a set U whose elements $x \in U$ themselves form a model of ZF (under the given membership relation). Add the axiom that every set is a member of some universe (since the cardinal of a universe is inaccessible, this amounts to assuming the existence of many inaccessible cardinals). A category is now always described by sets; that is, it is a set of objects together with a set of morphisms with the added structures which we have already described axiomatically. If U is a universe then a category *within* U is a category whose set of objects is a member of the universe U. Thus a category within U is much like a small category. Moreover, the usual comprehension axiom of ZF allows us to form $\mathbf{Cat}_U$, the category of all categories within U. This will be a category within some larger universe.

There are a number of variants of this idea. In Gabriel [**3**] and Sonner [**19**] a universe is described not as a model of ZF, but as a set closed under union, power set, and several more elementary operations. The essential point is that one may always form the functor category $\mathbf{A}^{\mathbf{B}}$ of two given categories—in general, by going to a larger universe U'. All these large and superlarge categories are sets, subject to the familiar manipulations of set theory. Given any universe U', one can always form the category of all categories within U'. This is still not that will-of-the-wisp, the category of all categories *überhaupt*. There are more subtle questions of the following sort: If R is a ring within the universe U and U' is some larger universe, how is the category of all left R-modules which are elements of U related to the category of all left R-modules which are elements of U'? One would like the relation to be close, but little seems to be known on this point.

Instead of using universes, set-theorists may prefer to work with the familiar levels V_α (all sets of rank less than α) of the cumulative hierarchy of sets (see e.g. Kreisel-Krivine [**8**]). Both the approach by small and large categories and the approach via universes can evidently be described within these levels. The **cat**s multiply; there appears to be a category $\mathbf{Cat}_\alpha$ whose objects are all categories **C** in which the set of morphisms is an element of V_α.

5. Basic questions. Our fundamental observation is just this: There is an appreciable body of results about categories (a few indications are given below, in §7) but the received methods of defining categories in terms of sets do not provide any *one* single context (i.e. any one model of a standard set theory) within which one can comprehensively state these results. Using universes, all the functor categories are there, but there is no category of all groups. Using Gödel-Bernays, one has the category of all (small) groups, but only a few functor categories. This situation raises a technical question and a philosophical question. The technical question is that of rearranging categories or sets (or both) to get this body of theorems whole. This question was explicitly formulated in Mac Lane [**15**]. Then Lawvere in [**10**] and [**11**] raised the basic possibility that the attempted explication of categories (and of other Mathematical objects) by sets might be replaced by an explication in terms of some other "fundamental" concepts. In detail, he proposed first an axiomatization of sets not in terms of the usual membership relation but in terms of the category of sets, and he provided an elementary such axiomatization. Going further, he proposed a list of axioms on **Cat** (the category of all categories) as a potential foundation of Mathematics. This involves both technical difficulties (Isbell [**7**]) and some exciting possibilities.

This also suggests a general philosophical question: Why should the "official" foundations of Mathematics necessarily be couched in terms of set-theory axiomatized by a membership relation? This perhaps inconvenient question, once clearly faced, raises the inescapable possibility of many alternative views of foundations. For instance, "function" rather than "set" might be taken as the fundamental notion. Both combinatory logic and von Neumann's original version of Gödel-Bernays may be regarded as axiomatizations of "functions"; there is good reason to suppose that other and more efficient such axiomatizations exist. Foundations in terms of sets make the most "primitive" Mathematical notion the starting point; there is considerable reason to suppose that the foundation would fit the facts better if it started with some more highly structured notions (set and function, category, or Mathematical structure). The set-theoretic approach is often described in terms of the intuitively constructed hierarchy V_α of cumulative types; one can easily provide similar hierarchies built on different structural concepts. In these ways, many alternative directions for investigation of the foundations are opened.

The prior situation in the foundations of Mathematics had in one respect a very simple structure. One could produce one formal system, say Zermelo-Fraenkel set theory, with the property that all the ordinary operations of practising Mathematicians could be carried out within this one system and on objects of this system. Explicitly, this meant that "every" Mathematical object was or could be defined to be a set, and that all of the arguments about these objects could be reduced to the axioms of ZF set theory. In practise, this did not mean that the arguments were actually so reduced; the working Mathematicians usually thought in terms of a naive set theory (probably one more or less equivalent to ZF) and was trained in describing everything as a set of this naive sort. This one-formal-system "monolithic" approach has also been convenient for specialists on foundations ever since Frege and Whitehead-Russell. On the one hand, all the classical nineteenth century problems of foundations (the construction of integers, real numbers, analysis; the properties of ordinal numbers; the axiom of choice and the continuum hypotheses) could be stated in this one system. On the other hand, alternative formal system could then be tested by comparison (as to strength or relative consistency) with this one system.

This happy situation no longer applies to the practice of category theory. Here the Mathematician is working with a variety of objects (categories of all groups, functor categories) which cannot all be described simultaneously as objects of any one foundational system. What follows? One might hope for some one new foundational system (the category of categories?) within which all the desired objects live; a practical requirement could be that this system could be used "naively" by Mathematicians not sophisticated in foundational research. The alternative would be the simultaneous use of several formal systems. This alternative is only foreshadowed by the Grothendieck use of universes. If the alternative is taken seriously, it has drastic consequences; the next section will give only a very small sample.

6. Multiple systems for foundations. Consider a formal foundation of Mathematics which is *multiple* in the sense that there are a number of different basic axiom systems T, T' which are collectively such that the objects of concern to the working Mathematician can be interpreted in one or the other of these systems. Recall also that the working mathematician usually regards the objects of his study as if they were really *there* (in some realistic or platonistic sense). "Really there" could be read as objects within a fixed interpretation M of one of the basic axiom systems. Here "interpretation" has the intuitive force of "model" (but do not take "model" in the usual technical sense of a model within some universe of sets). For example, this suggests that one would speak of a set-system M, meaning *an* interpretation (model) of ZF in exactly the same way that one speaks of an "abstract group" as a model of the axioms of group theory. Inevitably, there will be discussion of pseudo-set-systems and quasi-set-systems satisfying a variety of weaker axioms or different axioms (some not stated in terms of $\in$). The dogma that there is just *one* set theory disappears.

To handle categories, one must speak of "all so and so's" in some M. The language necessary to do this might be a very simple two sorted language with sorts *items* x, y, z (for things in M) and *classes* A, B, C (for collections of items). Primitive notions might be ordered pairs $\langle x, y\rangle$ of items and membership $x \in A$

(only for an item x in a class A). Axioms should include the expected axioms on the equality of ordered pairs, extensionality for classes, the existence of the null class, the singleton $\{x\}$, and the cartesian product $A \times B$ of classes, as well as a comprehension axiom for classes. These axioms can be regarded as simple axioms for a rudimentary fragment of class (or set) theory. In this language a category (such as a category of all so-and-so's in M) may be described as a class of objects and a class of morphisms with the usual structural properties. For such categories, we may then readily carry out the operations usual in elementary category theory; for example, one may construct the opposite of a category $\mathbf{C}^{\mathrm{op}}$ (reverse the direction of all morphisms) and the cartesian product $\mathbf{C} \times \mathbf{D}$ of two categories. More advanced constructions, such as that of functor categories, would require more powerful axioms (say, for each class B, the existence of a right adjoint to $A \mapsto A \times B$). The essential point is that the original system T, with model M, has been expanded by adjoining classes, in a way analogous to the passage from Zermelo-Fraenkel to Gödel-Bernays set theory. This passage illustrates what we mean by multiple systems of foundations, with varied notions of category (small, large, or otherwise) within these systems.

7. The uses of large categories. We turn from speculation to a few indications of the utility of extra large categories.

The Yoneda Lemma (Freyd [**2**, p. 112], Mac Lane [**14**, p. 54]) has become a fundamental tool of categorists. Let the large category $\mathbf{C}$ be locally small, so that each hom (A, B) is a set. Hold the object A fixed; then the function sending each object to the set hom (A, B) can be made into a functor from $\mathbf{C}$ to the (large) category $\mathbf{S}$ of (small) sets. This functor is usually written

$$h_A = \text{hom}(A, -): \mathbf{C} \to \mathbf{S}$$

and is called the covariant hom functor. Now let $F: \mathbf{C} \to \mathbf{S}$ be any other functor, and consider the class Nat (h_A, F) of all natural transformations $\theta: h_A \to F$. By definition, each such transformation assigns to every object B of $\mathbf{C}$ a morphism $\theta_B: \text{hom}(A, B) \to F(B)$ in the category $\mathbf{S}$. In particular, the function $\theta_A: \text{hom}(A, A) \to F(A)$ sends the identity morphism 1_A onto an element $\theta_A(1_A) \in F(A)$. This correspondence $\theta \mapsto \theta_A(1_A)$ is itself a function

$$\psi: \text{Nat}(h_A, F) \to F(A). \tag{1}$$

An easy argument shows that this function ψ is a bijection (is one-one onto). In fact the Yoneda lemma states simply that ψ is a bijection and a natural transformation.

The statement that ψ is natural means intuitively that the definition we have given for ψ depends on no artificial choices. It should also be interpreted as a natural transformation between functors. Indeed, if we regard the ordered pair (A, F) of the object A and the functor F as an object of the product category $\mathbf{C} \times \mathbf{S}^{\mathbf{C}}$, then both sides $F(A)$ and Nat (h_A, F) of (1) can be regarded as the values on (A, F) of suitable functors $\mathbf{C} \times \mathbf{S}^{\mathbf{C}} \to \mathbf{S}$; it is then routine to check that ψ as we have defined it is a natural transformation between these functors. The only trouble is that the functor category $\mathbf{S}^{\mathbf{C}}$ used here is not legitimate (for $\mathbf{C}$ large). If we employ universes instead of classes, the category $\mathbf{S}$ does not stay put.

Completeness is a basic property of a category, and is defined by closure under products and equalizers. First we define these terms. If f, $g: A \to B$ are two morphisms with the same domain A and the same codomain B, an *equalizer* of f, g is a morphism $e: R \to A$ such that (i) $fe = ge$; (ii) whenever $h: C \to A$ has $fh = gh$, there is a unique $h': C \to R$ such that $h = eh'$. This description clearly makes e (the opposite of) a universal. Let I be a set and A_i for $i \in I$ an I-indexed family of objects of $\mathbf{C}$ (in other words, A is a function on I to the objects of $\mathbf{C}$). A *product* of the family $\{A_i \mid i \in I\}$ consists of an object P of $\mathbf{C}$ and an I-indexed family $p_i: P \to A_i$, $i \in I$, of morphisms of $\mathbf{C}$ with the following (opposite of a) universal property: Whenever $f_i: C' \to A_i$ for $i \in I$, there is a unique $f: C \to P$ with $p_i f = f_i$ for all $i \in I$. For example, in the category of sets, the usual cartesian product ΠA_i of the sets A_i, equipped with its projections $p_i: \Pi A_i \to A_i$ to each of the factors, is readily seen to be a product in the categorical sense we have described. The corresponding cartesian products are also such products in the category of all small groups or of all small topological spaces.

A large category $\mathbf{C}$ is said to be *complete* whenever (a) it contains an equalizer for any two of its morphisms $f, g: A \to B$; (b) it contains a product for any set-indexed family of its objects. The importance of these two conditions is that together they imply that $\mathbf{C}$ contains (inverse) limits for all set-indexed systems (directed or not); technically, if $\mathbf{J}$ is any small category, every functor $F: \mathbf{J} \to \mathbf{C}$ has a limit (see Freyd [**2**], where "limit" is called "left root"). The discussion also indicates that the categories of sets, of groups, and of topological spaces (like many similar categories) are all complete. Observe that the set-class distinction enters vitally here. Completeness is defined using sets (of indices) by way of products or limits over sets; the familiar examples of complete categories are *large* categories hence are classes (see Freyd [**2**, p. 78]).

Completeness is a vital assumption in the fundamental existence theorem for adjoint functors due to Freyd. Let $\mathbf{C}$ be a complete locally small category and $U: \mathbf{C} \to \mathbf{X}$ a functor which carries products and equalizers (in $\mathbf{C}$) to products and equalizers, respectively, in $\mathbf{X}$. Then U has a left-adjoint F *provided* U satisfies the following "solution-set condition": For each object X of $\mathbf{X}$ there is a set J, a J-indexed family of objects R_j of $\mathbf{C}$ and a J-indexed family of morphisms $m_j: X \to UR_j$ of $\mathbf{X}$, such that to each $f: X \to UA$ there is a $j \in J$ and an $f': R_j \to A$ such that $f = U(f')m_j: X \to UA$. (Note that this solution-set condition is like the definition of an adjoint by a universal morphism, except that it specifies a *set* m_j of such morphisms which are "collectively" weakly universal.) For the proof of this adjoint functor theorem (with some needless added hypotheses) see Freyd [**2**, p. 84].

This and related theorems raise the question: If a given category $\mathbf{C}$ is not complete, can it be embedded in a complete category? Now $\mathbf{C}$ has a natural (Yoneda) embedding

$$Y: \mathbf{C} \to \mathbf{S}^{\mathbf{C}^{\mathrm{op}}}, \qquad Y(A) = h^A$$

into a functor category, where $\mathbf{C}^{\mathrm{op}}$ is the opposite of $\mathbf{C}$ and h^A is the contravariant hom-functor, with $h^A(B) = \hom(B, A)$. The completeness of $\mathbf{S}$ readily implies that $\mathbf{S}^{\mathbf{C}^{\mathrm{op}}}$ is complete, at least for $\mathbf{C}$ small. Hence a natural way to complete $\mathbf{C}$ is to embed $\mathbf{C}$ via Y in the closure of $Y(\mathbf{C})$ under products and equalizers. This procedure, however, is not immediately applicable if $\mathbf{C}$ is large. The best current theorems in this direction (Isbell [7]) make sophisticated use of set-theoretical considerations.

To illustrate the use of **Cat** we discuss fibered categories. Consider for instance the category of all (small) modules, where by a module we mean a pair (R, A) with R a ring, and A a left R-module. A morphism $(R, A) \to (R', A')$ of modules is a pair (s, f) consisting of a ring homomorphism $s: R \to R'$ and a homomorphism $f: A \to A'$ of (additive) abelian groups such that $f(ra) = (sr)(fa)$ for all $r \in R$, $a \in A$. Clearly this gives a category, and the assignments $(R, A) \mapsto R$, $(s, f) \mapsto s$ give a functor P on this category to the category of rings. We call P a fibered category, and observe that for any one ring R the fiber $P^{-1}(R)$ over R is itself a category.

In general a fibered category is a functor $P: \mathbf{A} \to \mathbf{C}$ satisfying certain lifting conditions about "cleavages", too complex to state here (see Gray, [4]). Since P is a functor, we have for each object C of $\mathbf{C}$ a category, the *fiber over* C, consisting of all objects A with $P(A) = C$ and all the morphisms f with $P(f) = 1_C$. Call this category $F(C)$. The cleavage conditions suffice to insure that each morphism $f: C \to C'$ on the "base" C yields a functor $F(f): F(C') \to F(C)$ backwards between the corresponding fibers; moreover, F is a (contravariant) functor. Indeed, F is a functor $\mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$. Effective treatment of the theory of fibered categories requires a systematic use of these functors to **Cat**, where **Cat** ought to be at least the category of all large categories. For this—and for many similar cases such as Benabou's "profunctors"—we need a working foundation which will handle the category of large categories.

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