Homework 1

1. The objects of **Rel** are sets, and an arrow $f : A \to B$ is a relation from A to B, that is, a subset $f \subseteq A \times B$. The equality relation $\{\langle a, a\rangle \in A \times A |\ a \in A\}$ is the identity arrow on a set A. Composition in **Rel** is to be given by

$$g \circ f = \{\langle a, c\rangle \in A \times C \mid \exists b\, (\langle a, b\rangle \in f \ \&\ \langle b, c\rangle \in g)\}$$

for $f \subseteq A \times B$ and $g \subseteq B \times C$.

Show that **Rel** is a category. Show also that there is a functor G : **Sets** $\to$ **Rel** taking objects to themselves and each function $f : A \to B$ to its graph,

$$G(f) = \{\langle a, f(a)\rangle \in A \times B \mid a \in A\}.$$

2. Consider the following isomorphisms of categories and determine which hold.
   (a) $\mathbf{Rel} \cong \mathbf{Rel}^{\mathrm{op}}$
   (b) $\mathbf{Sets} \cong \mathbf{Sets}^{\mathrm{op}}$
   (c) For a fixed set X with powerset $P(X)$, as poset categories $P(X) \cong P(X)^{\mathrm{op}}$ (the arrows in $P(X)$ are subset inclusions $A \subseteq B$ for all $A, B \subseteq X$).

3. (a) Show that in **Sets**, the isomorphisms are exactly the bijections.
   (b) Show that in **Monoids**, the isomorphisms are exactly the bijective homomorphisms.
   (c) Show that in **Posets**, the isomorphisms are *not* the same as the bijective homomorphisms.

4. Construct the "coslice category" $C/\mathbf{C}$ of a category $\mathbf{C}$ under an object C from the "slice category" operation $\mathbf{C}/C$ and the "dual category" operation $\mathbf{C}^{\mathrm{op}}$.

5. How many free categories on graphs are there which have exactly six arrows? Draw the graphs that generate these categories.

6. * Show that the free monoid functor

$$M : \mathbf{Sets} \to \mathbf{Mon}$$

exists, in two different ways:

(a) Assume the particular choice $M(X) = X^*$ and define its effect

$$M(f) : M(A) \to M(B)$$

on a function $f : A \to B$ to be

$$M(f)(a_1 \dots a_k) = f(a_1) \dots f(a_k), \qquad a_1, \dots a_k \in A.$$

(b) Assume only the UMP of the free monoid and use it to determine M on functions, showing the result to be a functor.