Homework 10

1. Let $\mathbf{C} \simeq \mathbf{D}$ be equivalent categories. Show that $\mathbf{C}$ has binary products if and only if $\mathbf{D}$ does. Consider: what sorts of properties of categories do *not* respect equivalence? Find one that respects isomorphism, but not equivalence.

2. (a) Let $\mathbf{C}$ be any category and $\mathbf{D}$ any complete category. Show that the functor category $\mathbf{D}^{\mathbf{C}}$ is also complete.
   (b) Use duality to show that the same is true for cocompleteness in place of completeness.
   (c) Conclude that categories of diagrams $\mathbf{Sets}^{\mathbf{C}}$ are complete and cocomplete.

3. Let $\mathbf{C}$ be a locally small category with binary products. Show that the Yoneda embedding
$$y\colon \mathbf{C} \to \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$$
preserves them. (Hint: this involves only a few lines of calculation.)

4. (a) Let $\mathbf{C}$ be a locally small, cartesian closed category. Use the Yoneda embedding to show that for any objects A, B, C in $\mathbf{C}$:
$$(A \times B)^C \cong A^C \times B^C$$
   (b) Show that if $\mathbf{C}$ also has binary coproducts, then:
$$A^{(B+C)} \cong A^B \times A^C$$

5. * Let $\mathbf{C}$ be a small category. Prove that the representable functors *generate* the diagram category $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$, in the following sense: given any objects $P, Q \in \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ and natural transformations $\varphi, \psi : P \to Q$, if for every representable functor yC and natural transformation $\vartheta : yC \to P$ one has $\varphi \circ \vartheta = \psi \circ \vartheta$, then $\varphi = \psi$. Thus the arrows in $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ are determined by their effect on generalized elements based at representables.