Homework 11

1. Let $\omega = \{0 \leq 1 \leq 2 \leq 3 \leq \ldots\}$, interpreted as (discrete) time. Consider the category $\mathbf{Sets}^{\omega}$ of "sets through time". Prove that this is a topos, and compute the subobject classifier explicitly.

2. For any set I with powerset $P(I)$, consider the functor $i : P(I) \to \mathbf{Sets}/I$ that takes a subset $U \subseteq I$ to its inclusion function $i(U) : U \to I$. Show that this is indeed a functor and that is has a left adjoint
$$\sigma : \mathbf{Sets}/I \longrightarrow P(I).$$
Does it have a right adjoint (in general)? Determine the units and counits.

3. Given a function $f : A \to B$ between sets, show that the direct image operation $\mathrm{im}(f) : P(A) \to P(B)$ is left adjoint to the inverse image $f^{-1} : P(B) \to P(A)$ (as functors between poset categories).

4. Given an object C in a category $\mathbf{C}$ with finite limits, show that the evident forgetful functor from the slice category $\mathbf{C}/C$,
$$U : \mathbf{C}/C \to \mathbf{C}$$
has a right adjoint. When does it have a left adjoint?

5. Any category $\mathbf{C}$ determines a preorder $P(\mathbf{C})$ by setting: $A \leq B$ if and only if there is an arrow $A \to B$. Show that the functor P is (left? right?) adjoint to the evident inclusion functor of preorders into categories. Determine the units and counits.

6. Show that the contravariant powerset functor $\mathcal{P} : \mathbf{Sets}^{\mathrm{op}} \to \mathbf{Sets}$ is self-adjoint.

7. * Show that there is a string of four adjoints between $\mathbf{Cat}$ and $\mathbf{Sets}$,
$$V \dashv F \dashv U \dashv R$$
where $U : \mathbf{Cat} \to \mathbf{Sets}$ is the forgetful functor to the set of objects $U(\mathbf{C}) = \mathbf{C}_0$. (Hint: for V, consider the "connected components" of a category.)