Homework 12

1. (a) A *coHeyting algebra* is a poset P such that P^{op} is a Heyting algebra. Determine the coHeyting implication operation a/b in a lattice L by adjointness (with respect to joins), and show that any Boolean algebra is a coHeyting algebra by explicitly defining this operation a/b in terms of the usual Boolean ones.

   (b) In a coHeyting algebra, there are operations of *coHeyting negation* $\sim p = 1/p$ and *coHeyting boundary* $\partial p = p \wedge \sim p$. State logical rules of inference for these operations.

   (c) A *biHeyting algebra* is a lattice that is both Heyting and coHeyting. Give an example of a biHeyting algebra that is not Boolean. (Hint: consider the lower sets in a poset.)

2. Let $\mathbb{T}$ be the equational theory with one constant symbol and one unary function symbol (no axioms). In any category with a terminal object, a natural numbers object (NNO) is just an initial $\mathbb{T}$-model. Show that

$$(\mathbb{N}, 0 \in \mathbb{N}, n+1 : \mathbb{N} \to \mathbb{N})$$

   is a NNO in **Sets**, and that any NNO is uniquely isomorphic to it (as a $\mathbb{T}$-model).

3. (a) Let $\mathbf{C}$ be a category and $T : \mathbf{C} \to \mathbf{C}$ an endofunctor. A *T-algebra* consists of an object A and an arrow $a : TA \to A$ in $\mathbf{C}$. A morphism $h : (a, A) \to (b, B)$ of T-algebras is a morphism $h : A \to B$ in $\mathbf{C}$ such that $h \circ a = b \circ T(h)$.

$$\begin{tikzcd} TA \arrow[r, "Th"] \arrow[d, "a"'] & TB \arrow[d, "b"] \\ A \arrow[r, "h"'] & B \end{tikzcd}$$

   Let $\mathbf{C}$ be a category with a terminal object 1 and binary coproducts. Let $T : \mathbf{C} \to \mathbf{C}$ be the endofunctor

$$T(C) = 1 + C.$$

Show that the category of T-algebras in this sense is equivalent to the category of $\mathbb{T}$-algebras (i.e. models) for the algebraic theory $\mathbb{T}$ with just one nullary and one unary operation,

$$T\text{-}\mathbf{Alg} \simeq \mathbb{T}\text{-}\mathbf{Alg}.$$

Conclude that free T-algebras exist in **Sets**, and that an initial T-algebra is the same thing as an NNO.

4. ("Lambek's Lemma") Show that for any endofunctor $T : \mathbf{C} \to \mathbf{C}$, if $i : TI \to I$ is an initial T-algebra, then i is an isomorphism.

   Hint: Consider a diagram of the following form, with suitable arrows.

$$\begin{tikzcd} TI \arrow[r] \arrow[d] & T^2 I \arrow[r] \arrow[d] & TI \arrow[d] \\ I \arrow[r] & TI \arrow[r] & I \end{tikzcd}$$

   Conclude that for any NNO N in any category (with 1 and $+$), there is an isomorphism $N + 1 \cong N$.

5. * (Lawvere's Hyperdoctrine Diagram)
   Recall the inclusion functor $i : P(I) \to \mathbf{Sets}/I$ that takes a subset $U \subseteq I$ to its inclusion function $i(U) : U \to I$. We know from last week that this functor has a left adjoint,

$$\sigma : \mathbf{Sets}/I \longrightarrow P(I),$$

   taking any $\alpha : A \to I$ to its image

$$\sigma(\alpha) = \{\alpha(a) \in I \mid a \in A\} \subseteq I.$$

   Given any function

$$f : J \to I$$

   consider the following diagram of functors.

$$\begin{tikzcd}[column sep=huge, row sep=large]
\mathbf{Sets}/I \arrow[r, shift left=3, "\Pi_f"'] \arrow[r, shift right=3, "\Sigma_f"'] \arrow[d, shift right, "\sigma_J"'] & \mathbf{Sets}/J \arrow[l, "f^*"'] \arrow[d, shift right, "\sigma_I"'] \\
P(J) \arrow[u, shift right, "i_J"'] \arrow[r, shift left=3, "\forall_f"'] \arrow[r, shift right=3, "\exists_f"'] & P(I) \arrow[l, "f^{-1}"'] \arrow[u, shift right, "i_I"']
\end{tikzcd}$$

There are adjunctions

$$\sigma \dashv i$$

(for both J and I), as well as

$$\Sigma_f \dashv f^* \dashv \Pi_f$$

and

$$\exists_f \dashv f^{-1} \dashv \forall_f$$

where $f^* : \mathbf{Sets}/I \to \mathbf{Sets}/J$ is pullback, and $f^{-1} : P(I) \to P(J)$ is inverse image.

Consider, which of the 6 possible squares involving either i or σ and corresponding pairs of f^{-1} and f^*, etc., commute.