Homework 2

1. Let A be a set. Define an *A-monoid* to be a monoid M equipped with a function $m : A \to U(M)$ (to the underlying set of M). A morphism $h : (M, m) \to (N, n)$ of A-monoids is to be a monoid homomorphism $h : M \to N$ such that $U(h) \circ m = n$ (a commutative triangle). Together with the evident identities and composites, this defines a category A-**Mon** of A-monoids.

   Show that an initial object in A-**Mon** is the same thing as a free monoid $M(A)$ on A. (Hint: compare their respective UMPs.)

2. Show that a function between sets is an epimorphism if it is surjective. Conclude that the isos in **Sets** are exactly the epi-monos.

3. With regard to a commutative triangle,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

   in any category $\mathbf{C}$, show

   a. if f and g are isos (resp. monos, resp. epis), so is h;
   b. if h is monic, so is f;
   c. if h is epic, so is g;
   d. (by example) if h is monic, g need not be.

4. Show that a homomorphism $h : G \to H$ of graphs is monic just if it is injective on both edges and vertices.

5. Show that in any category, any retract of a projective object is also projective.

6. In any category with binary products, show directly from the UMP that:

$$A \times (B \times C) \cong (A \times B) \times C.$$

7. * Show that the epis among posets are the surjections (on elements), and that the one-element poset **1** is projective. Show that sets, regarded as discrete posets, are projective in the category of posets. Give an example of a poset that is not projective. Show that every projective poset is discrete, i.e. a set. Conclude that **Sets** is (isomorphic to) the "full subcategory" of projectives in **Pos**, consisting of all projective posets and all monotone maps between them.