Homework 3

1. (a) For any index set I, define the product $\prod_{i \in I} X_i$ of an I-indexed family of objects $(X_i)_{i \in I}$ in a category, by giving a UMP generalizing that for binary products (the case $I = 2$).

   (b) Show that in **Sets**, for any set X the set X^I of all functions $f : I \to X$ has this UMP, with respect to the "constant family" where $X_i = X$ for all $i \in I$, and thus

   $$X^I \cong \prod_{i \in I} X$$

   (c) Show that $X \mapsto X^I$ is a functor $(-)^I : \mathbf{Sets} \to \mathbf{Sets}$.

2. In the category of types $\mathbf{C}(\lambda)$ of the λ-calculus, verify explicitly that there is a product functor $A, B \mapsto A \times B$ (i.e. using the λ-calculus, rather than the UMP of the product). Also show that, for any fixed type A, there is a functor $A \to (-) : \mathbf{C}(\lambda) \to \mathbf{C}(\lambda)$, taking any type X to the type $A \to X$.

3. Show that the forgetful functor $U : \mathbf{Mon} \to \mathbf{Sets}$ from monoids to sets is representable. Infer that U preserves all (small) products.

4. In any category $\mathbf{C}$, show that

   $$A \xrightarrow[c_1]{} C \xleftarrow[c_2]{} B$$

   is a coproduct diagram just if for every object Z, the map

   $$\begin{aligned} \mathrm{Hom}(C, Z) &\longrightarrow \mathrm{Hom}(A, Z) \times \mathrm{Hom}(B, Z) \\ f &\longmapsto \quad \langle f \circ c_1,\ f \circ c_2 \rangle \end{aligned}$$

   is an isomorphism. Do this by using duality and the corresponding fact about products, which you may take as given.

5. * Show in detail that the free monoid functor M preserves coproducts: for any sets A, B:

   $$M(A) + M(B) \cong M(A + B) \qquad \text{(canonically)}$$

   Do this as indicated in the text by using the UMPs of the coproducts $A + B$ and $M(A) + M(B)$ and of free monoids.