Homework 5

1. Prove that every monoid in the category of groups is an internal group. (Hint: use the Eckmann-Hilton argument.)

2. Let **C** be a category with pullbacks.

   (a) Show that an arrow $m : M \to X$ in **C** is monic if and only if the diagram below is a pullback.

$$\begin{tikzcd} M \arrow[r, "1_M"] \arrow[d, "1_M"'] & M \arrow[d, "m"] \\ M \arrow[r, "m"'] & X \end{tikzcd}$$

   Thus as an object in $\mathbf{C}/X$, m is monic iff $m \times m \cong m$.

   (b) Show that the pullback along an arrow $f : Y \to X$ of a pullback square over X,

$$\begin{tikzcd} A \times_X B \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & X \end{tikzcd}$$

   is again a pullback square over Y. (Hint: draw a cube and use the 2-pullbacks Lemma). Conclude that the pullback functor f^* preserves products.

   (c) Conclude from the foregoing that in a pullback square

$$\begin{tikzcd} M' \arrow[r] \arrow[d, "m'"'] & M \arrow[d, "m"] \\ A' \arrow[r, "f"'] & A \end{tikzcd}$$

   if m is monic, then so is m'.

3. (Pushouts)

   (a) Dualize the definition of a pullback to define the "copullback" (called the "pushout") of two arrows with common domain.

   (b) Indicate how to construct pushouts using coproducts and coequalizers (proof "by duality").

   (c) What is the pushout in posets of the two maps $0, 1 : \{*\} \to [0, 1]$, where $\{*\}$ is a one-element poset, and $[0, 1]$ is the unit interval.

4. *(Partial maps) For any category $\mathbf{C}$ with pullbacks, define the category $\mathbf{Par}(\mathbf{C})$ of partial maps in $\mathbf{C}$ as follows: the objects are the same as those of $\mathbf{C}$, but an arrow $f : A \to B$ is a pair $(|f|, U_f)$ where $U_f \rightarrowtail A$ is a subobject (an equivalence class of monomorphisms) and $|f| : U_f \to B$ (take a suitably-defined equivalence class of arrows), as indicated in the diagram:

$$\begin{tikzcd} U_f \arrow[r, "|f|"] \arrow[d, tail] & B \\ A & \end{tikzcd}$$

   Composition of $(|f|, U_f) : A \to B$ and $(|g|, U_g) : B \to C$ is given by taking a pullback and then composing to get $(|g \circ f|, |f|^*(U_g)$, as suggested by the follow diagram.

$$\begin{tikzcd} {|f|^*(U_g)} \arrow[r] \arrow[d, tail] & U_g \arrow[r, "|g|"'] \arrow[d, tail] & C \\ U_f \arrow[r, "|f|"'] \arrow[d, tail] & B & \\ A & & \end{tikzcd}$$

   Check to see that this really does define a category.