Homework 6

1. (Equalizers by pullbacks and products) Show that a category with pullbacks and products has equalizers as follows: given arrows $f, g : A \to B$, take the pullback indicated below, where $\Delta = \langle 1_B, 1_B \rangle$:

$$\begin{tikzcd} E \arrow[r] \arrow[d, "e"'] & B \arrow[d, "\Delta"] \\ A \arrow[r, "\langle f,g \rangle"'] & B \times B \end{tikzcd}$$

   Show that $e : E \to A$ is the equalizer of f and g.

2. Suppose the category $\mathbf{C}$ has limits of type $\mathbf{J}$, for some index category $\mathbf{J}$. For diagrams F and G of type $\mathbf{J}$ in $\mathbf{C}$, a morphism of diagrams $\theta : F \to G$ consists of arrows $\theta_i : Fi \to Gi$ for each $i \in \mathbf{J}$ such that for each $\alpha : i \to j$ in $\mathbf{J}$, one has $\theta_j F(\alpha) = G(\alpha)\theta_i$ (a commutative square). This makes $\mathbf{Diagrams}(\mathbf{J}, \mathbf{C})$ into a category (check this).

   Show that taking the vertex-objects of limiting cones determines a functor:

$$\varprojlim_{\mathbf{J}} : \mathbf{Diagrams}(\mathbf{J}, \mathbf{C}) \to \mathbf{C}$$

   Conclude that for any set I, there is a product functor,

$$\prod_{i \in I} : \mathbf{Sets}^I \to \mathbf{Sets}$$

   for I-indexed families of sets $(A_i)_{i \in I}$, generalizing the binary product functor $\times : \mathbf{Sets} \times \mathbf{Sets} \to \mathbf{Sets}$.

3. (a) Consider the sequence of posets $[0] \to [1] \to [2] \to \dots$, where

$$[n] = \{0 \leq \cdots \leq n\},$$

   and the arrows $[n] \to [n+1]$ are the evident inclusions. Determine the limit and colimit posets of this sequence.

(b) Do the same for the sequence of powerset boolean algebras,

$$\mathcal{P}(0) \leftarrow \mathcal{P}(1) \leftarrow \mathcal{P}(2) \leftarrow \ldots,$$

where the maps are determined by inverse image along the inclusions $0 \subseteq 1 \subseteq 2 \subseteq \ldots$.

4. Consider sequences of monoids,

$$M_0 \to M_1 \to M_2 \to \ldots$$
$$N_0 \leftarrow N_1 \leftarrow N_2 \leftarrow \ldots$$

and the following limits and colimits, constructed in the category of monoids:

$$\varinjlim_n M_n, \quad \varprojlim_n M_n, \quad \varinjlim_n N_n, \quad \varprojlim_n N_n.$$

(a) Suppose all M_n and N_n are abelian groups. Determine whether each of the four (co)limits $\varinjlim_n M_n$ etc. is also an abelian group.

(b) Suppose all M_n and N_n are finite groups. Determine whether each of the four (co)limits $\varinjlim_n M_n$ etc. has the following property: for every element x there is a number k such that $x^k = 1$ (the least such k is called the *order* of x).

5. * Show that the forgetful functor $U : \mathbf{Mon} \to \mathbf{Sets}$ creates all limits.