Homework 8

1. Complete the proof from the text of Kripke completeness for the positive fragement of IPC as follows:

   (a) Show that for any poset I, the exponential poset $\mathbf{2}^I$ is a Heyting algebra. (Hint: the limits and colimits are "pointwise", and the Heyting implication $p \Rightarrow q$ is defined at $i \in I$ by $(p \Rightarrow q)(i) = \top$ iff $p(j) \leq q(j)$ for all $j \geq i$.

   (b) Show that for any poset CCC $\mathbf{A}$, the map $y : \mathbf{A} \to \mathbf{2}^{\mathbf{A}^{\mathrm{op}}}$ defined in the text is indeed (i) monotone, (ii) injective, and (iii) preserves CCC structure.

2. Verify the claim in the text that the products $A \times B$ in categories $\mathbf{Sets}^I$ of I-indexed sets (I a poset) can be computed "pointwise",

   $$(A \times B)_i \;=\; A_i \times B_i$$

   Show, moreover, that the same is true for all limits.

3. Let I be a poset and G a "Kripke model" over I of the theory of monoids, i.e. a monoid in the category $\mathbf{Sets}^I$ of I-indexed sets. Show that G is the same thing as a functor $G : I \to \mathbf{Mon}$.

4. Consider the forgetful functors:

   $$\mathbf{Groups} \xrightarrow{U} \mathbf{Monoids} \xrightarrow{V} \mathbf{Sets}$$

   Say whether each is faithful, full, injective on objects, surjective on objects, injective on arrows, surjective on arrows.

5. * Recall that a *reflexive domain* is a model of the λ-theory with one basic type D, two terms $s : (D \to D) \to D$ and $r : D \to (D \to D)$, and one equation $r(s(x)) = x : D$. Verify that all reflexive domains in $\mathbf{Sets}$ are trivial, in the sense that for arbitrary terms $t, t' : D$, one has $[\![t]\!] = [\![t']\!]$ when interpreted in the domain.