Homework 9

1. Show that a functor category $\mathbf{D}^{\mathbf{C}}$ has binary products if $\mathbf{D}$ does (construct the product of two functors F and G "objectwise": $(F \times G)(C) = F(C) \times G(C)$).

2. Show that the map of sets

$$\begin{aligned} \eta_A : A &\longrightarrow PP(A) \\ a &\longmapsto \{U \subseteq A | a \in U\} \end{aligned}$$

   is the component at A of a natural transformation $\eta : 1_{\mathbf{Sets}} \to PP$, where $P : \mathbf{Sets}^{\mathrm{op}} \to \mathbf{Sets}$ is the (contravariant) power-set functor.

3. Let $\mathbf{C}$ be a locally small category. Show that there is a functor

$$\hom : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Sets}$$

   such that for each object C of $\mathbf{C}$,

$$\hom(C, -) : \mathbf{C} \to \mathbf{Sets}$$

   is the covariant representable functor and

$$\hom(-, C) : \mathbf{C}^{\mathrm{op}} \to \mathbf{Sets}$$

   is the contravariant one. (*Hint: use the Bifunctor Lemma*)

4. (a) Complete the proof that, for any set I, the category of I-indexed families of sets, regarded as the functor category $\mathbf{Sets}^I$, is equivalent to the slice category $\mathbf{Sets}/I$ of sets over I.

$$\mathbf{Sets}^I \;\simeq\; \mathbf{Sets}/I$$

   (b) * Show that reindexing of families along a function $f : J \to I$, given by precomposition,

$$\mathbf{Sets}^f((A_i)_{i \in I}) = (A_{f(j)})_{j \in J}$$

is represented by pullback, in the sense that the following diagram of categories and functors commutes up to natural isomorphism.

$$
\begin{array}{ccc}
\mathbf{Sets}^I & \xrightarrow{\simeq} & \mathbf{Sets}/I \\
{\scriptstyle \mathbf{Sets}^f}\downarrow & & \downarrow{\scriptstyle f^*} \\
\mathbf{Sets}^J & \xrightarrow[\simeq]{} & \mathbf{Sets}/J
\end{array}
$$

Here $f^* : \mathbf{Sets}/I \to \mathbf{Sets}/J$ is the pullback functor along the function $f : J \to I$.