5

LIMITS AND COLIMITS

In this chapter we first briefly discuss some topics—namely subobjects and pullbacks—relating to the definitions that we already have. This is partly in order to see how these are used, but also because we will need this material soon. Then we approach things more systematically, defining the general notion of a limit, which subsumes many of the particular abstract characterizations we have met so far. Of course, there is a dual notion of colimit, which also has many interesting applications. After a brief look at one more elementary notion in the next chapter, we shall go on to what may be called "higher category theory."

5.1 Subobjects

We have seen that every subset $U \subseteq X$ of a set X occurs as an equalizer and that equalizers are always monomorphisms. So it is natural to regard monos as generalized subsets. That is, a mono in **Groups** can be regarded as a subgroup, a mono in **Top** as a subspace, and so on.

The rough idea is this: given a monomorphism,

$$m : M \rightarrowtail X$$

in a category **G** of structured sets of some sort—call them "gadgets"—the image sub*set*

$$\{m(y) \mid y \in M\} \subseteq X$$

which may be written $m(M)$, is often a sub-gadget of X to which M is isomorphic via m.

$$m : M \xrightarrow{\sim} m(M) \subseteq X$$

More generally, we can think of the mono $m : M \rightarrowtail X$ itself as determining a "part" of X, even in categories that do not have underlying functions to take images of.

Definition 5.1. A *subobject* of an object X in a category **C** is a monomorphism,

$$m : M \rightarrowtail X.$$

Given subobjects m and m' of X, a morphism $f : m \to m'$ is an arrow in $\mathbf{C}/X$, as in:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, "m"'] & M' \arrow[d, "m'"] \\ & X \end{tikzcd}$$

Thus we have a category,

$$\mathrm{Sub}_{\mathbf{C}}(X)$$

of subobjects of X in $\mathbf{C}$.

In this definition, since m' is monic, there is at most one f as in the diagram above, so that $\mathrm{Sub}_{\mathbf{C}}(X)$ is a preorder category. We define the relation of *inclusion* of subobjects by:

$$m \subseteq m' \quad \text{iff} \quad \text{there exists some } f : m \to m'$$

Finally, we say that m and m' are *equivalent*, written $m \equiv m'$, if and only if they are isomorphic as subobjects, that is, $m \subseteq m'$ and $m' \subseteq m$. This holds just if there are f and f' making both triangles below commute.

$$\begin{tikzcd} M \arrow[r, shift right, "f"'] \arrow[dr, "m"'] & M' \arrow[l, shift right, "f'"'] \arrow[d, "m'"] \\ & X \end{tikzcd}$$

Observe that, in the above diagram, $m = m'f = mf'f$, and since m is monic, $f'f = 1_M$ and similarly $ff' = 1_{M'}$. So $M \cong M'$ via f. Thus we see that *equivalent subobjects have isomorphic domains*. We sometimes abuse notation and language by calling M the subobject when the mono $m : M \rightarrowtail X$ is clear.

Remark 5.2. It is often convenient to pass from the preorder

$$\mathrm{Sub}_{\mathbf{C}}(X)$$

to the *poset* given by factoring out the equivalence relation "$\equiv$". Then a subobject is an equivalence class of monos under mutual inclusion.

In **Sets**, under this notion of subobject, one then has an isomorphism,

$$\mathrm{Sub}_{\mathbf{Sets}}(X) \cong P(X)$$

that is, every subobject is represented by a unique subset. We shall use both notions of subobject, making clear when monos are intended, and when equivalence classes thereof are intended.

Note that if $M' \subseteq M$ then the arrow f which makes this so in

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr] & M' \arrow[d] \\ & X \end{tikzcd}$$

is also monic, so also M' is a subobject of M. Thus we have a functor

$$\mathrm{Sub}(M') \to \mathrm{Sub}(X)$$

defined by composition with f (since the composite of monos is monic).

In terms of generalized elements of an object X,

$$z : Z \to X$$

one can define a *local membership relation*,

$$z \in_X M$$

between such elements and subobjects $m : M \rightarrowtail X$ by

$$z \in_X M \;\text{ iff }\; \text{there exists } f : Z \to M \text{ such that } z = mf$$

Since m is monic, if z factors through it then it does so uniquely.

Example 5.3. An equalizer

$$\begin{tikzcd} E \arrow[r] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

is a subobject of A with the property

$$z \in_A E \quad \text{iff } f(z) = g(z)$$

Thus, we can regard E as the subobject of generalized elements $z : Z \to A$ such that $f(z) = g(z)$, suggestively:

$$E = \{z \in Z \mid f(z) = g(z)\} \subseteq A$$

In categorical logic, one develops a way of making this intuition even more precise by giving a calculus of such subobjects.

5.2 Pullbacks

The notion of a pullback, like that of a product, is one that comes up very often in mathematics and logic. It is a generalization of both intersection and inverse image.

We begin with the definition.

Definition 5.4. In any category $\mathbf{C}$, given arrows f, g with $\operatorname{cod}(f) = \operatorname{cod}(g)$,

$$\begin{tikzcd} & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

the *pullback* of f and g consists of arrows

$$\begin{tikzcd} P \arrow[r, "p_2"'] \arrow[d, "p_1"] & B \\ A & \end{tikzcd}$$

such that $fp_1 = gp_2$ and universal with this property. That is, given any $z_1 : Z \to A$ and $z_2 : Z \to B$ with $fz_1 = gz_2$, there exists a unique $u : Z \to P$ with $z_1 = p_1 u$ and $z_2 = p_2 u$. The situation is indicated in the following diagram.

$$\begin{tikzcd} Z \arrow[dr, dotted, "u"] \arrow[drr, "z_2"] \arrow[ddr, "z_1"'] & & \\ & P \arrow[r, "p_2"'] \arrow[d, "p_1"] & B \arrow[d, "g"] \\ & A \arrow[r, "f"'] & C \end{tikzcd}$$

One sometimes uses product-style notation for pullbacks.

$$
\begin{tikzcd}
Z \arrow[dr, "\langle z_1, z_2\rangle"] \arrow[drr, bend left, "z_2"] \arrow[ddr, bend right, "z_1"'] & & \\
& A \times_C B \arrow[r, "p_2"'] \arrow[d, "p_1"] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}
$$

Pullbacks are clearly unique up to isomorphism since they are given by a UMP (universal mapping property). Here, this means that given two pullbacks of a given pair of arrows, the uniquely determined maps between the pullbacks are mutually inverse.

In terms of generalized elements, any $z \in A \times_C B$, can be written uniquely as $z = \langle z_1, z_2 \rangle$ with $f z_1 = g z_2$. This makes

$$A \times_C B = \{\langle z_1, z_2\rangle \in A \times B \mid f z_1 = g z_2\}$$

look like a subobject of $A \times B$, determined as an equalizer of $f \circ \pi_1$ and $g \circ \pi_2$. In fact, this is so.

Proposition 5.5. *In a category with products and equalizers, given a corner of arrows*

$$
\begin{tikzcd}
& B \arrow[d, "g"] \\
A \arrow[r, "f"'] & C
\end{tikzcd}
$$

Consider the diagram

$$\begin{tikzcd}
E \arrow[dr, "e"] \arrow[drr, "p_2"] \arrow[ddr, "p_1"'] & & \\
& A \times B \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

in which e is an equalizer of $f\pi_1$ and $g\pi_2$ and $p_1 = \pi_1 e$, $p_2 = \pi_2 e$. Then E, p_1, p_2 is a pullback of f and g. Conversely, if E, p_1, p_2 are given as such a pullback, then the arrow

$$e = \langle p_1, p_2 \rangle : E \to A \times B$$

is an equalizer of $f\pi_1$ and $g\pi_2$.

Proof. Take

$$\begin{tikzcd}
Z \arrow[r, "z_2"] \arrow[d, "z_1"'] & B \\
A &
\end{tikzcd}$$

with $fz_1 = gz_2$. We have $\langle z_1, z_2 \rangle : Z \to A \times B$ so

$$f\pi_1\langle z_1, z_2 \rangle = g\pi_2\langle z_1, z_2 \rangle.$$

Thus, there is a $u : Z \to E$ to the equalizer with $eu = \langle z_1, z_2 \rangle$. Then

$$p_1 u = \pi_1 e u = \pi_1 \langle z_1, z_2 \rangle = z_1$$

and

$$p_2 u = \pi_2 e u = \pi_2 \langle z_1, z_2 \rangle = z_2.$$

If also $u' : Z \to E$ has $p_i u' = z_i, i = 1, 2$, then $\pi_i e u' = z_i$ so $eu' = \langle z_1, z_2 \rangle = eu$ whence $u' = u$ since e in monic. The converse is similar. □

Corollary 5.6. *If a category* **C** *has binary products and equalizers, then it has pullbacks.*

The foregoing gives an explicit construction of a pullback in **Sets** as a subset of the product:

$$\{\langle a, b\rangle \mid fa = gb\} = A \times_C B \hookrightarrow A \times B$$

Example 5.7. In **Sets**, take a function $f : A \to B$ and a subset $V \subseteq B$. Let, as usual,

$$f^{-1}(V) = \{a \in A \mid f(a) \in V\} \subseteq A$$

and consider

$$\begin{tikzcd} f^{-1}(V) \arrow[r, "\bar{f}"] \arrow[d, "j"'] & V \arrow[d, "i"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

where i and j are the canonical inclusions and $\bar{f}$ is the evident factorization of the restriction of f to $f^{-1}(V)$ (since $a \in f^{-1}(V) \Rightarrow f(a) \in V$).

This diagram is a pullback (observe that $z \in f^{-1}(V) \Leftrightarrow fz \in V$ for all $z : Z \to A$). Thus, the inverse image

$$f^{-1}(V) \subseteq A$$

is *determined uniquely up to isomorphism* as a pullback.

As suggested by the previous example, we can use pullbacks to *define* inverse images in categories other than **Sets**. Indeed, given a pullback in any category:

$$\begin{tikzcd} A \times_B M \arrow[r] \arrow[d, "m'"'] & M \arrow[d, "m"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

if m is monic, then m' is monic. (Exercise!)

Thus we see that, for fixed $f : A \to B$, taking pullbacks induces a map

$$f^{-1} : \mathrm{Sub}(B) \to \mathrm{Sub}(A)$$
$$m \mapsto m'$$

We will show that f^{-1} also respects equivalence of subobjects,

$$M \equiv N \Rightarrow f^{-1}(M) \equiv f^{-1}(N)$$

by showing that f^{-1} is a functor; that is our next goal.

5.3 Properties of pullbacks

We start with the following simple lemma, which seems to come up all the time.

Lemma 5.8. *(Two-pullbacks) Consider the commutative diagram below in a category with pullbacks:*

$$
\begin{tikzcd}
F \arrow[r, "f'"] \arrow[d, "h''"'] & E \arrow[r, "g'"] \arrow[d, "h'"] & D \arrow[d, "h"] \\
A \arrow[r, "f"'] & B \arrow[r, "g"'] & C
\end{tikzcd}
$$

1. *If the two squares are pullbacks, so is the outer rectangle. Thus,*

$$A \times_B (B \times_C D) \cong A \times_C D$$

2. *If the right square and the outer rectangle are pullbacks, so is the left square.*

Proof. Diagram chase. □

Corollary 5.9. *The pullback of a commutative triangle is a commutative triangle. Specifically, given a commutative triangle as on the right end of the following "prism diagram"*

$$
\begin{tikzcd}
A' \arrow[rr, "h_\alpha"'] \arrow[dd, "\alpha'"'] \arrow[dr, dotted, "\gamma'"] & & A \arrow[dd, "\alpha"'] \arrow[dr, "\gamma"] & \\
& B' \arrow[rr, "h_\beta"'] \arrow[dl, "\beta'"] & & B \arrow[dl, "\beta"] \\
C' \arrow[rr, "h"'] & & C &
\end{tikzcd}
$$

for any $h : C' \to C$, if one can form the pullbacks α' and β' as on the left end, then there exists a unique γ' as indicated, making the left end a commutative triangle, and the upper face a commutative rectangle, and indeed a pullback.

Proof. Apply the two-pullbacks lemma. □

Proposition 5.10. *Pullback is a functor. That is, for fixed $h : C' \to C$ in a category $\mathbf{C}$ with pullbacks, there is a functor*

$$h^* : \mathbf{C}/C \to \mathbf{C}/C'$$

defined by

$$(A \xrightarrow{\alpha} C) \mapsto (C' \times_C A \xrightarrow{\alpha'} C')$$

where α' is the pullback of α along h, and the effect on an arrow $\gamma : \alpha \to \beta$ is given by the foregoing corollary.

Proof. One must check that

$$h^*(1_X) = 1_{h^*X}$$

and

$$h^*(g \circ f) = h^*(g) \circ h^*(f)$$

These can easily be verified by repeated applications of the two-pullbacks lemma. For example, for the first condition, consider

$$\begin{tikzcd}
A' \arrow[r, "h'"'] \arrow[d, "1_{A'}"'] & A \arrow[d, "1_A"] \\
A' \arrow[r, "h'"'] \arrow[d, "\alpha'"'] & A \arrow[d, "\alpha"] \\
C' \arrow[r, "h"'] & C
\end{tikzcd}$$

If the lower square is a pullback, then plainly so is the outer rectangle, whence the upper square is, too, and we have

$$h^*1_X = 1_{X'} = 1_{h^*X}.$$

□

Corollary 5.11. *Let $\mathbf{C}$ be a category with pullbacks. For any arrow $f : A \to B$ in $\mathbf{C}$ we have the following diagram of categories and functors:*

$$\begin{tikzcd}
\mathrm{Sub}(A) \arrow[d] & \mathrm{Sub}(B) \arrow[l, "f^{-1}"'] \arrow[d] \\
\mathbf{C}/A & \mathbf{C}/B \arrow[l, "f^*"]
\end{tikzcd}$$

This commutes simply because f^{-1} is defined to be the restriction of f^* to the subcategory $\mathrm{Sub}(B)$*. Thus, in particular, f^{-1} is functorial:*

$$M \subseteq N \;\Rightarrow\; f^{-1}(M) \subseteq f^{-1}(N)$$

It follows that $M \equiv N$ implies $f^{-1}(M) \equiv f^{-1}(N)$, so that f^{-1} is also defined on equivalence classes.

$$f^{-1}/{\equiv} : \mathrm{Sub}(B)/{\equiv} \longrightarrow \mathrm{Sub}(A)/{\equiv}$$

Example 5.12. Consider a pullback in **Sets**:

$$\begin{tikzcd} E \arrow[r, "f'"] \arrow[d, "g'"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

We saw that

$$E = \{\langle a, b\rangle \mid f(a) = g(b)\}$$

can be constructed as an equalizer

$$\begin{tikzcd} E \arrow[r, "\langle f', g'\rangle"] & A \times B \arrow[r, shift left, "f\pi_1"] \arrow[r, shift right, "g\pi_2"'] & C \end{tikzcd}$$

Now let $B = 1$, $C = 2 = \{\top, \bot\}$, and $g = \top : 1 \to 2$. Then the equalizer

$$\begin{tikzcd} E \arrow[r] & A \times 1 \arrow[r, shift left, "f\pi_1"] \arrow[r, shift right, "\top\pi_2"'] & 2 \end{tikzcd}$$

is how we already described the "extension" of the "propositional function" $f : A \to 2$. Therefore we can rephrase the correspondence between subsets $U \subseteq A$ and their characteristic functions $\chi_U : A \to 2$ in terms of pullbacks:

$$\begin{tikzcd} U \arrow[r, "!"] \arrow[d] & 1 \arrow[d, "\top"] \\ A \arrow[r, "\chi_U"'] & 2 \end{tikzcd}$$

Precisely, the isomorphism,

$$2^A \cong P(A)$$

given by taking a function $\varphi : A \to 2$ to its "extension"

$$V_\varphi = \{x \in A \mid \varphi(x) = \top\}$$

can be described as a pullback.

$$V_\varphi = \{x \in A \mid \varphi(x) = \top\} = \varphi^{-1}(\top)$$

Now suppose we have any function

$$f : B \to A$$

and consider the induced inverse image operation

$$f^{-1} : P(A) \to P(B)$$

given by pullback, as in example 5.9 above. Taking the extension $V_\varphi \subseteq A$, consider the two-pullback diagram

$$\begin{tikzcd} f^{-1}(V_\varphi) \arrow[r] \arrow[d] & V_\varphi \arrow[r] \arrow[d] & 1 \arrow[d, "\top"] \\ B \arrow[r, "f"'] & A \arrow[r, "\varphi"'] & 2 \end{tikzcd}$$

We therefore have (by the two-pullbacks lemma)

$$f^{-1}(V_\varphi) = f^{-1}(\varphi^{-1}(\top)) = (\varphi f)^{-1}(\top) = V_{\varphi f}$$

which from a logical point of view expresses the fact that the substitution of a term f for the variable x in the propositional function φ is modeled by taking the pullback along f of the corresponding extension

$$f^{-1}(\{x \in A \mid \varphi(x) = \top\}) = \{y \in B \mid \varphi(f(y)) = \top\}.$$

Note that we have shown that for any function $f : B \to A$ the following square commutes

$$\begin{tikzcd} 2^A \arrow[r, "\cong"] \arrow[d, "2^f"'] & P(A) \arrow[d, "f^{-1}"] \\ 2^B \arrow[r, "\cong"'] & P(B) \end{tikzcd}$$

where $2^f : 2^A \to 2^B$ is precomposition $2^f(g) = g \circ f$. In a situation like this, one says that the isomorphism

$$2^A \cong P(A)$$

is *natural* in A, which is obviously a much stronger condition than just having isomorphisms at each object A. We will consider such "naturality" systematically later. It was in fact one of the phenomena that originally gave rise to category theory.

Example 5.13. Let I be an index set, and consider an I-indexed family of sets:

$$(A_i)_{i\in I}$$

Given any function $\alpha : J \to I$, there is a J-indexed family

$$(A_{\alpha(j)})_{j\in J}\ ,$$

obtained by "reindexing along α." This reindexing can also be described as a pullback. Specifically, for each set A_i take the constant, i-valued function $p_i : A_i \to I$ and consider the induced map on the coproduct

$$p = [p_i] : \coprod_{i\in I} A_i \to I$$

The reindexed family $(A_{\alpha(j)})_{j\in J}$ can be obtained by taking a pullback along α, as indicated in the following diagram:

$$\begin{tikzcd} \coprod_{j\in J} A_{\alpha(j)} \arrow[r] \arrow[d, "q"'] & \coprod_{i\in I} A_i \arrow[d, "p"] \\ J \arrow[r, "\alpha"'] & I \end{tikzcd}$$

where q is the indexing projection for $(A_{\alpha(j)})_{j\in J}$ analogous to p. In other words, we have

$$J \times_I (\coprod_{i\in I} A_i) \ \cong\ \coprod_{j\in J} A_{\alpha(j)}$$

The reader should work out the details as an instructive exercise.

5.4 Limits

We have already seen that the notions of product, equalizer, and pullback are not independent; the precise relation between them is this.

Proposition 5.14. *A category has finite products and equalizers iff it has pullbacks and a terminal object.*

Proof. The "only if" direction has already been done. For the other direction, suppose $\mathbf{C}$ has pullbacks and a terminal object 1.

- For any objects A, B we clearly have $A \times B \cong A \times_1 B$, as indicated in the following:

$$\begin{tikzcd} A \times B \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & 1 \end{tikzcd}$$

- For any arrows $f, g : A \to B$, the equalizer $e : E \to A$ is constructed as the following pullback:

$$\begin{tikzcd} E \arrow[r, "h"] \arrow[d, "e"'] & B \arrow[d, "\Delta = \langle 1_B, 1_B \rangle"] \\ A \arrow[r, "\langle f, g \rangle"'] & B \times B \end{tikzcd}$$

In terms of generalized elements,

$$E = \{(a, b) \mid \langle f, g \rangle(a) = \Delta b\}$$

where $\langle f, g \rangle(a) = \langle fa, ga \rangle$ and $\Delta(b) = \langle b, b \rangle$. So,

$$\begin{aligned} E &= \{\langle a, b \rangle \mid f(a) = b = g(a)\} \\ &\cong \{a \mid f(a) = g(a)\} \end{aligned}$$

which is just what we want. An easy diagram chase shows that

$$\begin{tikzcd} E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

is indeed an equalizer. □

Product, terminal object, pullback, and equalizer, are all special cases of the general notion of a *limit*, which we will consider now. First, we need some preliminary definitions.

Definition 5.15. Let $\mathbf{J}$ and $\mathbf{C}$ be categories. A *diagram* of *type* $\mathbf{J}$ in $\mathbf{C}$ is a functor.

$$D : \mathbf{J} \to \mathbf{C}.$$

We will write the objects in the "index category" $\mathbf{J}$ lower case, $i, j, \ldots$ and the values of the functor $D : \mathbf{J} \to \mathbf{C}$ in the form D_i, D_j, etc.

A *cone* to a diagram D consists of an object C in $\mathbf{C}$ and a family of arrows in $\mathbf{C}$,

$$c_j : C \to D_j$$

one for each object $j \in J$, such that for each arrow $\alpha : i \to j$ in $\mathbf{J}$, the following triangle commutes.

$$\begin{tikzcd} C \arrow[r, "c_j"] \arrow[d, "c_i"'] & D_j \\ D_i \arrow[ur, "D_\alpha"'] & \end{tikzcd}$$

A *morphism* of cones

$$\vartheta : (C, c_j) \to (C', c'_j)$$

is an arrow ϑ in $\mathbf{C}$ making each triangle,

$$\begin{tikzcd} C \arrow[r, "\vartheta"] \arrow[dr, "c_j"'] & C' \arrow[d, "c'_j"] \\ & D_j \end{tikzcd}$$

commute. That is, such that $c_j = c'_j \circ \vartheta$ for all $j \in \mathbf{J}$. Thus, we have an evident category

$$\mathbf{Cone}(D)$$

of cones to D.

We are here thinking of the diagram D as a "picture of $\mathbf{J}$ in $\mathbf{C}$." A cone to such a diagram D is then imagined as a many-sided pyramid over the "base" D and a morphism of cones is an arrow between the apexes of such pyramids. (The reader should draw some pictures at this point!)

Definition 5.16. A *limit* for a diagram $D : \mathbf{J} \to \mathbf{C}$ is a terminal object in $\mathbf{Cone}(D)$. A *finite limit* is a limit for a diagram on a finite index category $\mathbf{J}$.

We often denote a limit in the form

$$p_i : \varprojlim_j D_j \to D_i.$$

Spelling out the definition, the limit of a diagram D has the following UMP: given any cone (C, c_j) to D, there is a unique arrow $u : C \to \varprojlim_j D_j$ such that for all j,

$$p_j \circ u = c_j.$$

Thus the limiting cone $(\varprojlim_j D_j, p_j)$ can be thought of as the "closest" cone to the diagram D, and indeed any other cone (C, c_j) comes from it just by composing with an arrow at the vertex, namely $u : C \to \varprojlim_j D_j$.

$$\begin{tikzcd}
C \arrow[rrr, dotted, "u"] \arrow[drr, "c_i"'] \arrow[drrrr] & & & \varprojlim_j D_j \arrow[dl] \arrow[dr, "p_j"] & \\
& & D_i \arrow[rr, "D_\alpha"'] & & D_j
\end{tikzcd}$$

Example 5.17. Take $\mathbf{J} = \{1, 2\}$ the discrete category with two objects and no nonidentity arrows. A diagram $D : \mathbf{J} \to \mathbf{C}$ is a pair of objects $D_1, D_2 \in \mathbf{C}$. A cone on D is an object of $\mathbf{C}$ equipped with arrows

$$D_1 \xleftarrow{c_1} C \xrightarrow{c_2} D_2.$$

And a limit of D is a terminal such cone, that is, a *product* in $\mathbf{C}$ of D_1 and D_2,

$$D_1 \xleftarrow{p_1} D_1 \times D_2 \xrightarrow{p_2} D_2.$$

Thus, in this case,

$$\varprojlim_j D_j \;\cong\; D_1 \times D_2.$$

Example 5.18. Take $\mathbf{J}$ to be the following category:

$$\begin{tikzcd}
\cdot \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & \cdot
\end{tikzcd}$$

A diagram of type $\mathbf{J}$ looks like

$$\begin{tikzcd}
D_1 \arrow[r, shift left, "D_\alpha"] \arrow[r, shift right, "D_\beta"'] & D_2
\end{tikzcd}$$

and a cone is a pair of arrows

$$\begin{tikzcd}
D_1 \arrow[r, shift left, "D_\alpha"] \arrow[r, shift right, "D_\beta"'] & D_2 \\
C \arrow[u, "c_1"] \arrow[ur, "c_2"'] &
\end{tikzcd}$$

such that $D_\alpha c_1 = c_2$ and $D_\beta c_1 = c_2$; thus, $D_\alpha c_1 = D_\beta c_1$. A limit for D is therefore an *equalizer* for D_α, D_β.

Example 5.19. If $\mathbf{J}$ is empty, there is just one diagram $D : \mathbf{J} \to \mathbf{C}$, and a limit for it is thus a *terminal object* in $\mathbf{C}$,

$$\varprojlim_{j \in \mathbf{0}} D_j \;\cong\; 1.$$

Example 5.20. If $\mathbf{J}$ is the finite category

$$\begin{tikzcd} & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$

we see that a limit for a diagram of the form

$$\begin{tikzcd} & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

is just a pullback of f and g,

$$\varprojlim_{j} D_j \;\cong\; A \times_C B.$$

Thus, we have shown half of the following:

Proposition 5.21. *A category has all finite limits iff it has finite products and equalizers (resp. pullbacks and a terminal object by the last proposition).*

Here a category $\mathbf{C}$ is said to *have all finite limits* if every finite diagram $D : \mathbf{J} \to \mathbf{C}$ has a limit in $\mathbf{C}$.

Proof. We need to show that any finite limit can be constructed from finite products and equalizers. Take a finite diagram

$$D : \mathbf{J} \to \mathbf{C}.$$

As a first approximation, the product

$$\prod_{i \in \mathbf{J}_0} D_i \tag{5.1}$$

over the set $\mathbf{J}_0$ of objects at least has projections $p_j : \prod_{i \in \mathbf{J}_0} D_i \to D_j$ of the right sort. But these can't be expected to commute with the arrows $D_\alpha : D_i \to D_j$ in the diagram D, as they must. So, as in making a pullback from a product and an equalizer, we consider also the product $\prod_{(\alpha:i\to j)\in\mathbf{J}_1} D_j$ over all the arrows (the set $\mathbf{J}_1$), and two special maps,

$$\prod_i D_i \overset{\phi}{\underset{\psi}{\rightrightarrows}} \prod_{\alpha:i\to j} D_j$$

which record the effect of the arrows in the diagram on the product of the objects. Specifically, we define ϕ and ψ by taking their composites with the projections π_α from the second product to be, respectively:

$$\pi_\alpha \circ \phi = \phi_\alpha = \pi_{\mathrm{cod}(\alpha)}$$
$$\pi_\alpha \circ \psi = \psi_\alpha = D_\alpha \circ \pi_{\mathrm{dom}(\alpha)}$$

where $\pi_{\mathrm{cod}(\alpha)}$ and $\pi_{\mathrm{dom}(\alpha)}$ are projections from the first product.

Now, in order to get the subobject of the product 5.1 on which the arows in the diagram D commute, we take the equalizer:

$$E \xrightarrow{e} \prod_i D_i \overset{\phi}{\underset{\psi}{\rightrightarrows}} \prod_{\alpha:i\to j} D_j$$

We will show that (E, e_i) is a limit for D, where $e_i = \pi_i \circ e$. To that end, take any arrow $c : C \to \prod_i D_i$, and write $c = \langle c_i \rangle$ for $c_i = \pi_i \circ c$. Observe that the family of arrows $(c_i : C \to D_i)$ is a cone to D if and only if $\phi c = \psi c$. Indeed,

$$\phi\langle c_i\rangle = \psi\langle c_i\rangle$$

iff for all α,

$$\pi_\alpha\phi\langle c_i\rangle = \pi_\alpha\psi\langle c_i\rangle.$$

But,

$$\pi_\alpha\phi\langle c_i\rangle = \phi_\alpha\langle c_i\rangle = \pi_{\mathrm{cod}(\alpha)}\langle c_i\rangle = c_j$$

and

$$\pi_\alpha\psi\langle c_i\rangle = \psi_\alpha\langle c_i\rangle = D_\alpha \circ \pi_{\mathrm{dom}(\alpha)}\langle c_i\rangle = D_\alpha \circ c_i.$$

Whence $\phi c = \psi c$ iff for all $\alpha : i \to j$ we have $c_j = D_\alpha \circ c_i$ thus, iff $(c_i : C \to D_i)$ is a cone, as claimed. It follows that (E, e_i) is a cone, and that any cone $(c_i : C \to D_i)$ gives an arrow $\langle c_i\rangle : C \to \prod_i D_i$ with $\phi\langle c_i\rangle = \psi\langle c_i\rangle$, thus there is a unique factorization $u : C \to E$ of $\langle c_i\rangle$ through E, which is clearly a morphism of cones. □

Since we made no real use of the finiteness of the index category apart from the existence of certain products, essentially the same proof yields the following:

Corollary 5.22. *A category has all limits of some cardinality iff it has all equalizers and products of that cardinality, where* **C** *is said to* have limits *(resp.*

products*) of cardinality* κ *iff* $\mathbf{C}$ *has a limit for every diagram* $D : \mathbf{J} \to \mathbf{C}$ *where* $\mathrm{card}(\mathbf{J}_1) \leq \kappa$ *(resp.* $\mathbf{C}$ *has all products of* κ *many objects).*

The notions of cones and limits of course dualize to give those of *cocones* and *colimits.* One then has the following dual theorem.

Theorem 5.23. *A category* $\mathbf{C}$ *has finite colimits iff it has finite coproducts and coequalizers (resp. iff it has pushouts and an initial object).* $\mathbf{C}$ *has all colimits of size* κ *iff it has coequalizers and coproducts of size* κ*.*

5.5 Preservation of limits

Here is an application of the construction of limits by products and equalizers.

Definition 5.24. A functor $F : \mathbf{C} \to \mathbf{D}$ is said to *preserve limits of type* $\mathbf{J}$ if, whenever $p_j : L \to D_j$ is a limit for a diagram $D : \mathbf{J} \to \mathbf{C}$; the cone $Fp_j : FL \to FD_j$ is then a limit for the diagram $FD : \mathbf{J} \to \mathbf{D}$. Briefly,

$$F(\varprojlim D_j) \cong \varprojlim F(D_j).$$

A functor that preserves all limits is said to be *continuous.*

For example, let $\mathbf{C}$ be a locally small category with all small limits, such as posets or monoids. Recall the representable functor

$$\mathrm{Hom}(C, -) : \mathbf{C} \to \mathbf{Sets}$$

for any object $C \in \mathbf{C}$, taking $f : X \to Y$ to

$$f_* : \mathrm{Hom}(C, X) \to \mathrm{Hom}(C, Y)$$

where $f_*(g : C \to X) = f \circ g$.

Proposition 5.25. *The representable functors* $\mathrm{Hom}(C, -)$ *preserve all limits.*

Since limits in $\mathbf{C}$ can be constructed from products and equalizers, it suffices to show that $\mathrm{Hom}(C, -)$ preserves products and equalizers. (Actually, even if $\mathbf{C}$ does not have all limits, the representable functors will preserve those limits that do exist; we leave that as an exercise.)

Proof.

- $\mathbf{C}$ has a terminal object 1, for which,

$$\mathrm{Hom}(C, 1) = \{!_C\} \cong 1.$$

- Consider a binary product $X \times Y$ in $\mathbf{C}$. Then we already know that,

$$\mathrm{Hom}(C, X \times Y) \cong \mathrm{Hom}(C, X) \times \mathrm{Hom}(C, Y)$$

  by composing any $f : C \to X \times Y$ with the two product projections $p_1 : X \times Y \to X$, and $p_2 : X \times Y \to Y$.

- For arbitrary products $\prod_{i \in I} X_i$ one has analogously:

$$\mathrm{Hom}(C, \prod_i X_i) \cong \prod_i \mathrm{Hom}(C, X_i)$$

- Given an equalizer in $\mathbf{C}$,

$$E \xrightarrow[e]{} X \overset{f}{\underset{g}{\rightrightarrows}} Y$$

consider the resulting diagram,

$$\mathrm{Hom}(C, E) \xrightarrow[e_*]{} \mathrm{Hom}(C, X) \overset{f_*}{\underset{g_*}{\rightrightarrows}} \mathrm{Hom}(C, Y).$$

To show this is an equalizer in **Sets**, let $h : C \to X \in \mathrm{Hom}(C, X)$ with $f_* h = g_* h$. Then $fh = gh$, so there is a unique $u : C \to E$ such that $eu = h$. Thus, we have a unique $u \in \mathrm{Hom}(C, E)$ with $e_* u = eu = h$. So $e_* : \mathrm{Hom}(C, E) \to \mathrm{Hom}(C, X)$ is indeed the equalizer of f_* and g_*.

□

Definition 5.26. A functor of the form $F : \mathbf{C}^{\mathrm{op}} \to \mathbf{D}$ is called a *contravariant functor* on $\mathbf{C}$. Explicitly, such a functor takes $f : A \to B$ to $F(f) : F(B) \to F(A)$ and $F(g \circ f) = F(f) \circ F(g)$.

A typical example of a contravariant functor is a representable functor of the form,

$$\mathrm{Hom}_{\mathbf{C}}(-, C) : \mathbf{C}^{\mathrm{op}} \to \mathbf{Sets}$$

for any $C \in \mathbf{C}$ (where $\mathbf{C}$ is any locally small category). Such a contravariant representable functor takes $f : X \to Y$ to

$$f^* : \mathrm{Hom}(Y, C) \to \mathrm{Hom}(X, C)$$

by $f^*(g : X \to C) = g \circ f$.

The dual version of the foregoing proposition is then this:

Corollary 5.27. *Contravariant representable functors map all colimits to limits.*

For example, given a coproduct $X + Y$ in any locally small category $\mathbf{C}$, there is a canonical isomorphism,

$$\mathrm{Hom}(X + Y, C) \;\cong\; \mathrm{Hom}(X, C) \times \mathrm{Hom}(Y, C) \tag{5.2}$$

given by precomposing with the two coproduct inclusions.

From an example in Section 2.3 we can therefore conclude that the ultrafilters in a coproduct $A + B$ of Boolean algebras correspond exactly to pairs of ultrafilters (U, V), with U in A and V in B. This follows because we showed there that the ultrafilter functor $\mathrm{Ult} : \mathbf{BA}^{\mathrm{op}} \to \mathbf{Sets}$ is representable:

$$\mathrm{Ult}(B) \;\cong\; \mathrm{Hom}_{\mathbf{BA}}(B, 2).$$

Another case of the above iso (5.2) is the familiar law of exponents for sets:

$$C^{X+Y} \cong C^X \times C^Y$$

The arithmetical law of exponents $k^{m+n} = k^n \cdot k^m$ is actually a special case of this!

5.6 Colimits

Let us briefly discuss some special colimits, since we did not really say much about them in the foregoing section.

First, we consider *pushouts* in **Sets**. Suppose we have two functions

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & C \\ B & \end{tikzcd}$$

We can construct the pushout of f and g like this. Start with the coproduct (disjoint sum):

$$B \longrightarrow B + C \longleftarrow C$$

Now identify those elements $b \in B$ and $c \in C$ such that, for some $a \in A$,

$$f(a) = b \quad \text{and} \quad g(a) = c$$

That is, we take the equivalence relation $\sim$ on $B+C$ generated by the conditions $f(a) \sim g(a)$ for all $a \in A$.

Finally, we take the quotient by $\sim$ to get the pushout

$$(B + C)/\!\sim \quad \cong \quad B +_A C,$$

which can be imagined as B placed next to C, with the respective parts that are images of A "pasted together" or overlapping. This construction follows simply by dualizing the one for pullbacks by products and equalizers.

Example 5.28. Pushouts in **Top** are similarly formed from coproducts and coequalizers, which can be made first in **Sets** and then topologized as sum and quotient spaces. Pushouts are used e.g. to construct spheres from disks. Indeed, let D^2 be the (two-dimensional) disk and S^1 the one-dimensional sphere (i.e. the circle), with its inclusion $i : S^1 \to D^2$ as the boundary of the disk. Then the 2-sphere S^2 is the pushout,

$$\begin{tikzcd} S^1 \arrow[r, "i"] \arrow[d, "i"'] & D^2 \arrow[d] \\ D^2 \arrow[r] & S^2. \end{tikzcd}$$

Can you see the analogous construction of S^1 at the next lower dimension?

In general, a colimit for a diagram $D : \mathbf{J} \to \mathbf{C}$ is of course an initial object in the category of *cocones*. Explicitly, a *cocone from the base* D consists of an object C (the vertex) and arrows $c_j : D_j \to C$ for each $j \in \mathbf{J}$, such that for all $\alpha : i \to j$ in $\mathbf{J}$,

$$c_j \circ D(\alpha) = c_i$$

A morphism of cocones $f : (C, (c_j)) \to (C', (c_j{}'))$ is an arrow $f : C \to C'$ in $\mathbf{C}$ such that $f \circ c_j = c_j{}'$ for all $j \in \mathbf{J}$. An initial cocone is the expected thing: one that maps uniquely *to* any other cocone from D. We write such a colimit in the form:

$$\varinjlim_{j \in \mathbf{J}} D_j$$

Now let us consider some examples of a particular kind of colimit that comes up quite often, namely over a linearly ordered index category. Our first example is what is sometimes called a *direct limit* of a sequence of algebraic objects, say groups. A similar construction will work for any sort of algebras (but non-equational conditions are not always preserved by direct limits).

Example 5.29. *Direct limit of groups.* Suppose we are given a sequence,

$$G_0 \xrightarrow[g_0]{} G_1 \xrightarrow[g_1]{} G_2 \xrightarrow[g_2]{} \cdots$$

of groups and homomorphisms, and we want a "colimiting" group G_∞ with homomorphisms

$$u_n : G_n \to G_\infty$$

satisfying $u_{n+1} \circ g_n = u_n$. Moreover, G_∞ should be "universal" with this property. I think you can see the colimit setup here:

- the index category is the ordinal number $\omega = (\mathbb{N}, \leq)$, regarded as a poset category,
- the sequence

$$G_0 \xrightarrow[g_0]{} G_1 \xrightarrow[g_1]{} G_2 \xrightarrow[g_2]{} \cdots$$

is a diagram of type ω in the category **Groups**,

- the colimiting group is the colimit of the sequence:

$$G_\infty \cong \varinjlim_{n\in\omega} G_n$$

This group always exists, and can be constructed as follows. Begin with the coproduct (disjoint sum) of sets

$$\coprod_{n\in\omega} G_n.$$

Then make identifications $x_n \sim y_m$, where $x_n \in G_n$ and $y_m \in G_m$, to ensure in particular that

$$x_n \sim g_n(x_n)$$

for all $x_n \in G_n$ and $g_n : G_n \to G_{n+1}$.

This means, specifically, that the elements of G_∞ are equivalence classes of the form

$$[x_n], \quad x_n \in G_n$$

for any n, and $[x_n] = [y_m]$ iff for some $k \geq m, n$,

$$g_{n,k}(x_n) = g_{m,k}(y_m)$$

where, generally, if $i \leq j$, we define

$$g_{i,j} : G_i \to \cdots \to G_j$$

by composing consecutive g's as in $g_{i,j} = g_{j-1} \circ \ldots \circ g_i$. The reader can easily check that this is indeed the equivalence relation generated by all the conditions $x_n \sim g_n(x_n)$.

The operations on G_∞ are now defined by

$$[x] \cdot [y] = [x' \cdot y']$$

where $x \sim x'$, $y \sim y'$, and $x', y' \in G_n$ for n sufficiently large. The unit is just $[u_0]$, and we take,

$$[x]^{-1} = [x^{-1}].$$

One can easily check that these operations are well defined, and determine a group structure on G_∞, which moreover makes all the evident functions

$$u_n : G_n \to G_\infty\ , \qquad u_n(x) = [x]$$

into homomorphisms.

The universality of G_∞ and the u_n results from the fact that the construction is essentially a colimit in **Sets**, equipped with an induced group structure. Indeed, given any group H and homomorphisms $h_n : G_n \to H$ with $h_{n+1} \circ g_n = h_n$ define

$h_\infty : G_\infty \to H$ by $h_\infty([x_n]) = h_n(x_n)$. This is easily seen to be well defined and indeed a homomorphism. Moreover, it is the unique *function* that commutes with all the u_n.

The fact that the ω-colimit G_∞ of groups can be constructed as the colimit of the underlying sets is a case of a general phenomenon, expressed by saying that the forgetful functor $U : \mathbf{Groups} \to \mathbf{Sets}$ "creates ω-colimits."

Definition 5.30. A functor $F : \mathbf{C} \to \mathbf{D}$ is said to *create limits of type* $\mathbf{J}$ if for every diagram $C : \mathbf{J} \to \mathbf{C}$ and limit $p_j : L \to FC_j$ in $\mathbf{D}$ there is a unique cone $\overline{p_j} : \overline{L} \to C_j$ in $\mathbf{C}$ with $F(\overline{L}) = L$ and $F(\overline{p_j}) = p_j$, which, furthermore, is a limit for C. Briefly, every limit in $\mathbf{D}$ is the image of a unique cone in $\mathbf{C}$, which is a limit there. The notion of *creating colimits* is defined analogously.

In these terms, then, we have the following proposition, the remaining details of which have in effect already been shown.

Proposition 5.31. *The forgetful functor $U : \mathbf{Groups} \to \mathbf{Sets}$ creates ω-colimits. It also creates all limits.*

The same fact holds quite generally for other categories of algebraic objects, that is, sets equipped with operations satisfying some equations. Observe that not *all* colimits are created in this way. For instance, we have already seen (in example) that the coproduct of two abelian groups has their *product* as underlying set.

Example 5.32. *Cumulative hierarchy.* Another example of an ω-colimit is the "cumulative hierarchy" construction encountered in set theory. Let us set

$$\begin{aligned} V_0 &= \emptyset \\ V_1 &= \mathcal{P}(\emptyset) \\ &\vdots \\ V_{n+1} &= \mathcal{P}(V_n) \end{aligned}$$

Then there is a sequence of subset inclusions,

$$\emptyset = V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots$$

since, generally, $A \subseteq B$ implies $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ for any sets A and B. The colimit of the sequence

$$V_\omega = \varinjlim_n V_n$$

is called the *cumulative hierarchy* of rank ω. One can of course continue this construction through higher ordinals $\omega + 1, \omega + 2, \ldots$.

More generally, let us start with some set A (of "atoms"), and let

$$V_0(A) = A$$

and then put

$$V_{n+1}(A) = A + \mathcal{P}(V_n(A)),$$

i.e. the set of all elements *and* subsets of A. There is a sequence $V_0(A) \to V_1(A) \to V_2(A) \to \dots$ as follows. Let

$$v_0 : V_0(A) = A \to A + \mathcal{P}(A) = V_1(A)$$

be the left coproduct inclusion. Given $v_{n-1} : V_{n-1}(A) \to V_n(A)$, let $v_n : V_n(A) \to V_{n+1}(A)$ be defined by

$$v_n = 1_A + \mathcal{P}_!(v_{n-1}) : A + \mathcal{P}(V_{n-1}(A)) \to A + \mathcal{P}(V_n(A))$$

where $\mathcal{P}_!$ denotes the *covariant* powerset functor, taking a function $f : X \to Y$ to the "image under f" operation $\mathcal{P}_!(f) : \mathcal{P}(X) \to \mathcal{P}(Y)$, defined by taking $U \subseteq X$ to

$$\mathcal{P}_!(f)(U) = \{f(u) \mid u \in U\} \subseteq Y.$$

The idea behind the sequence is that we start with A, then add all the subsets of A, then add all the *new* subsets that can be formed from all of those elements, and so on. The colimit of the sequence

$$V_\omega(A) = \varinjlim_n V_n(A)$$

is called the *cumulative hierarchy* (of rank ω) over A. Of course, $V_\omega = V_\omega(\emptyset)$.

Now suppose we have some function

$$f : A \to B.$$

Then there is a map

$$V_\omega(f) : V_\omega(A) \to V_\omega(B),$$

determined by the colimit description of V_ω, as indicated in the following diagram.

$$\begin{tikzcd}
V_0(A) \arrow[r] \arrow[d, "f_0"'] & V_1(A) \arrow[r] \arrow[d, "f_1"] & V_2(A) \arrow[r] \arrow[d, "f_2"] & \dots \arrow[r] & V_\omega(A) \arrow[d, "f_\omega"] \\
V_0(B) \arrow[r] & V_1(B) \arrow[r] & V_2(B) \arrow[r] & \dots \arrow[r] & V_\omega(B)
\end{tikzcd}$$

Here the f_n are defined by

$$\begin{aligned}
f_0 &= f : A \to B, \\
f_1 &= f + \mathcal{P}_!(f) : A + \mathcal{P}(A) \to B + \mathcal{P}(B), \\
&\vdots \\
f_{n+1} &= f + \mathcal{P}_!(f_n) : A + \mathcal{P}(V_n(A)) \to B + \mathcal{P}(V_n(B)).
\end{aligned}$$

Since all the squares clearly commute, we have a cocone on the diagram of $V_n(A)$'s with vertex $V_\omega(B)$, and there is thus a unique $f_\omega : V_\omega(A) \to V_\omega(B)$ that completes the diagram.

Thus we see that the cumulative hierarchy is functorial.

Example 5.33. *ωCPOs.* An *ωCPO* is a poset that is "ω-cocomplete," meaning it has all colimits of type $\omega = (\mathbb{N}, \leq)$. Specifically, a poset D is an ωCPO if for every diagram $d : \omega \to D$, i.e. every chain of elements of D,

$$d_0 \leq d_1 \leq d_2 \leq \cdots$$

we have a colimit $d_\omega = \varinjlim d_n$. This is an element of D such that:

1. $d_n \leq d_\omega$ for all $n \in \omega$;
2. for all $x \in D$, if $d_n \leq x$ for all $n \in \omega$, then also $d_\omega \leq x$.

A monotone map of ωCPOs

$$h : D \to E$$

is called *continuous* if it preserves colimits of type ω, that is,

$$h(\varinjlim d_n) = \varinjlim h(d_n).$$

An application of these notions is the following:

Proposition 5.34. *If D is an ωCPO with initial element* 0 *and*

$$h : D \to D$$

is continuous, then h has a fixed point

$$h(x) = x$$

which, moreover, is least among all fixed points.

Proof. We use "Newton's method," which can be used, for example, to find fixed points of monotone, continuous functions $f : [0, 1] \to [0, 1]$. Consider the sequence $d : \omega \to D$, defined by

$$d_0 = 0$$

$$d_{n+1} = h(d_n)$$

Since $0 \leq d_0$, repeated application of h gives $d_n \leq d_{n+1}$. Now take the colimit $d_\omega = \varinjlim_{n\in\omega} d_n$. Then

$$\begin{aligned}
h(d_\omega) &= h(\varinjlim_{n\in\omega} d_n) \\
&= \varinjlim_{n\in\omega} h(d_n) \\
&= \varinjlim_{n\in\omega} d_{n+1} \\
&= d_\omega.
\end{aligned}$$

The last step follows because the first term $d_0 = 0$ of the sequence is trivial.

Moreover, if x is also a fixed point, $h(x) = x$, then we have

$$\begin{aligned}
d_0 &= 0 \leq x \\
d_1 &= h(0) \leq h(x) = x \\
&\vdots \\
d_{n+1} &= h(d_n) \leq h(x) = x.
\end{aligned}$$

So also $d_\omega \leq x$, since d_ω is the colimit. □

Finally, here is an example of how (co)limits depend on the ambient category. We consider colimits *of* posets and ωCPOs, rather than *in* them.

Let us define the finite ωCPOs

$$\omega_n = \{k \leq n \mid k \in \omega\}$$

then we have continuous inclusion maps:

$$\omega_0 \to \omega_1 \to \omega_2 \to \cdots$$

In **Pos**, the colimit exists, and is ω, as can be easily checked. *But* ω itself is not ω-complete. Indeed, the sequence

$$0 \leq 1 \leq 2 \leq \cdots$$

has no colimit. So the colimit of the ω_n in the category of ωCPOs, if it exists, must be something else. In fact it is $\omega + 1$.

$$0 \leq 1 \leq 2 \leq \cdots \leq \omega$$

For then any bounded sequence has a colimit in the bounded part, and any unbounded one has ω as colimit. The moral is that even ω-colimits are not always created in **Sets**, and indeed the colimit is sensitive to the ambient category in which it is taken.

5.7 Exercises

1. Show that a pullback of arrows

$$\begin{tikzcd} A \times_X B \arrow[r, "p_2"] \arrow[d, "p_1"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & X \end{tikzcd}$$

in a category $\mathbf{C}$ is the same thing as their product in the slice category $\mathbf{C}/X$.

2. Let $\mathbf{C}$ be a category with pullbacks.

   (a) Show that an arrow $m : M \to X$ in $\mathbf{C}$ is monic if and only if the diagram below is a pullback.

$$\begin{tikzcd} M \arrow[r, "1_M"] \arrow[d, "1_M"'] & M \arrow[d, "m"] \\ M \arrow[r, "m"'] & X \end{tikzcd}$$

   Thus as an object in $\mathbf{C}/X$, m is monic iff $m \times m \cong m$.

   (b) Show that the pullback along an arrow $f : Y \to X$ of a pullback square over X,

$$\begin{tikzcd} A \times_X B \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & X \end{tikzcd}$$

   is again a pullback square over Y. (Hint: draw a cube and use the 2-pullbacks Lemma). Conclude that the pullback functor f^* preserves products.

(c) Conclude from the foregoing that in a pullback square

$$\begin{tikzcd} M' \arrow[r] \arrow[d, "m'"'] & M \arrow[d, "m"] \\ A' \arrow[r, "f"'] & A \end{tikzcd}$$

if m is monic, then so is m'.

3. For any object A in a category $\mathbf{C}$ and any subobjects $M, N \in \mathrm{Sub}_{\mathbf{C}}(A)$, show $M \subseteq N$ iff for every generalized element $z : Z \to A$ (arbitrary arrow with codomain A):

$$z \in_A M \text{ implies } z \in_A N.$$

4. Show that in any category, given a pullback square

$$\begin{tikzcd} M' \arrow[r] \arrow[d, "m'"'] & M \arrow[d, "m"] \\ A' \arrow[r, "f"'] & A \end{tikzcd}$$

if m is monic, then so is m'.

5. For any object A in a category $\mathbf{C}$ and any subobjects $M, N \in \mathrm{Sub}_{\mathbf{C}}(A)$, show $M \subseteq N$ iff for every generalized element $z : Z \to A$ (arbitrary arrow with codomain A):

$$z \in_A M \text{ implies } z \in_A N.$$

6. (Equalizers by pullbacks and products) Show that a category with pullbacks and products has equalizers as follows: given arrows $f, g : A \to B$, take the pullback indicated below, where $\Delta = \langle 1_B, 1_B \rangle$:

$$\begin{tikzcd} E \arrow[r] \arrow[d, "e"'] & B \arrow[d, "\Delta"] \\ A \arrow[r, "\langle f, g \rangle"'] & B \times B \end{tikzcd}$$

Show that $e : E \to A$ is the equalizer of f and g.

7. Let $\mathbf{C}$ be a locally small category, and $D : \mathbf{J} \to \mathbf{C}$ any diagram for which a limit exists in $\mathbf{C}$. Show that for any object $C \in \mathbf{C}$, the representable functor

$$\mathrm{Hom}_{\mathbf{C}}(C, -) : \mathbf{C} \to \mathbf{Sets}$$

preserves the limit of D.

8. (Partial maps) For any category $\mathbf{C}$ with pullbacks, define the category $\mathbf{Par}(\mathbf{C})$ of partial maps in $\mathbf{C}$ as follows: the objects are the same as those of $\mathbf{C}$, but an arrow $f : A \to B$ is a pair $(|f|, U_f)$ where $U_f \rightarrowtail A$ is a subobject and $|f| : U_f \to B$ is a suitable equivalence class of arrows, as indicated in the diagram:

$$\begin{tikzcd} U_f \arrow[r, "|f|"] \arrow[d, tail] & B \\ A & \end{tikzcd}$$

Composition of $(|f|, U_f) : A \to B$ and $(|g|, U_g) : B \to C$ is given by taking a pullback and then composing to get $(|g \circ f|, |f|^*(U_g))$, as suggested by the follow diagram.

$$\begin{tikzcd} {|f|^*(U_g)} \arrow[r] \arrow[d, tail] & U_g \arrow[r, "|g|"'] \arrow[d, tail] & C \\ U_f \arrow[r, "|f|"'] \arrow[d, tail] & B & \\ A & & \end{tikzcd}$$

Verify that this really does define a category, and show that there is a functor,

$$\mathbf{C} \to \mathbf{Par}(\mathbf{C})$$

which is the identity on objects.

9. Suppose the category $\mathbf{C}$ has limits of type $\mathbf{J}$, for some index category $\mathbf{J}$. For diagrams F and G of type $\mathbf{J}$ in $\mathbf{C}$, a morphism of diagrams $\theta : F \to G$ consists of arrows $\theta_i : Fi \to Gi$ for each $i \in \mathbf{J}$ such that for each $\alpha : i \to j$ in $\mathbf{J}$, one has $\theta_j F(\alpha) = G(\alpha)\theta_i$ (a commutative square). This makes $\mathbf{Diagrams}(\mathbf{J}, \mathbf{C})$ into a category (check this).
Show that taking the vertex-objects of limiting cones determines a functor:

$$\varprojlim_{\mathbf{J}} : \mathbf{Diagrams}(J, \mathbf{C}) \to \mathbf{C}$$

Infer that for any set I, there is a product functor,

$$\prod_{i\in I} : \mathbf{Sets}^I \to \mathbf{Sets}$$

for I-indexed families of sets $(A_i)_{i\in I}$.

10. (Pushouts)
    (a) Dualize the definition of a pullback to define the "copullback" (usually called the "pushout") of two arrows with common domain.
    (b) Indicate how to construct pushouts using coproducts and coequalizers (proof "by duality").
11. Let $R \subseteq X \times X$ be an equivalence relation on a set X, with quotient $q : X \twoheadrightarrow Q$. Show that the following is an equalizer,

$$\mathcal{P}Q \xrightarrow{\mathcal{P}q} \mathcal{P}X \overset{\mathcal{P}r_1}{\underset{\mathcal{P}r_2}{\rightrightarrows}} \mathcal{P}R,$$

where $r_1, r_2 : R \rightrightarrows X$ are the two projections of $R \subseteq X$, and $\mathcal{P}$ is the (contravariant) powerset functor. (Hint: $\mathcal{P}X \cong 2^X$.)

12. Consider the sequence of posets $[0] \to [1] \to [2] \to \ldots$, where

$$[n] = \{0 \leq \cdots \leq n\},$$

and the arrows $[n] \to [n+1]$ are the evident inclusions. Determine the limit and colimit posets of this sequence.

13. Consider sequences of monoids,

$$M_0 \to M_1 \to M_2 \to \ldots$$
$$N_0 \leftarrow N_1 \leftarrow N_2 \leftarrow \ldots$$

and the following limits and colimits, constructed in the category of monoids:

$$\varinjlim_n M_n, \quad \varprojlim_n M_n, \quad \varinjlim_n N_n, \quad \varprojlim_n N_n.$$

   (a) Suppose all M_n and N_n are abelian groups. Determine whether each of the four (co)limits $\varinjlim_n M_n$ etc. is also an abelian group.
   (b) Suppose all M_n and N_n are finite groups. Determine whether each of the four (co)limits $\varinjlim_n M_n$ etc. has the following property: for every element x there is a number k such that $x^k = 1$ (the least such k is called the *order* of x).