6

EXPONENTIALS

We have now managed to unify most of the universal mapping properties that we have seen so far with the notion of limits (or colimits). Of course, the free algebras are an exception to this. In fact, it will turn out that there is a common source of such UMP's, but it lies somewhat deeper, in the notion of *adjoints*, which unify free algebras, limits, and other universals of various kinds.

Next we are going to look at one more elementary universal structure, which is also an example of a universal that is not a limit. This important structure is called an "exponential," and it can be thought of as a categorical notion of a "function space." As we'll see it subsumes much more than just that, however.

6.1 Exponential in a category

Let us start by considering a function of sets,

$$f(x, y) : A \times B \to C$$

written using variables x over A and y over B. If we now hold $a \in A$ fixed, we have a function

$$f(a, y) : B \to C$$

and thus an element

$$f(a, y) \in C^B$$

of the set of all such functions.

Letting a vary over A then gives a map, which I will write like this:

$$\tilde{f} : A \to C^B$$

defined by $a \mapsto f(a, y)$.

The map $\tilde{f} : A \to C^B$ takes the "parameter" a to the function $f_a(y) : B \to C$. It's uniquely determined by the equation

$$\tilde{f}(a)(b) = f(a, b).$$

Indeed, *any* map

$$\phi : A \to C^B$$

is uniquely of the form

$$\phi = \tilde{f}$$

for some $f : A \times B \to C$. For we can set

$$f(a, b) := \phi(a)(b).$$

What this means, in sum, is that we have an isomorphism of Hom-sets:

$$\mathrm{Hom}_{\mathbf{Sets}}(A \times B, C) \cong \mathrm{Hom}_{\mathbf{Sets}}(A, C^B)$$

That is, there is a bijective correspondence between functions of the form $f : A \times B \to C$ and those of the form $\tilde{f} : A \to C^B$, which we can display schematically thus:

$$\frac{f : A \times B \to C}{\tilde{f} : A \to C^B}$$

This bijection is mediated by a certain operation of *evaluation*, which we have indicated in the foregoing by using variables. In order to generalize the indicated bijection to other categories, we're going to need to make this evaluation operation explicit, too.

In **Sets**, it is the function

$$\mathrm{eval} : C^B \times B \to C$$

defined by $(g, b) \mapsto g(b)$, that is,

$$\mathrm{eval}(g, b) = g(b).$$

This evaluation function has the following UMP: given any set A and any function

$$f : A \times B \to C$$

there is a unique function

$$\tilde{f} : A \to C^B$$

such that $\mathrm{eval} \circ (\tilde{f} \times 1_B) = f$. That is,

$$\mathrm{eval}(\tilde{f}(a), b) = f(a, b). \tag{6.1}$$

Here is the diagram:

$$\begin{tikzcd}
C^B & C^B \times B \arrow[r, "\mathrm{eval}"] & C \\
A \arrow[u, "\tilde{f}"] & A \times B \arrow[u, "\tilde{f} \times 1_B"] \arrow[ur, "f"'] &
\end{tikzcd}$$

You can read the equation (6.1) off from this diagram by taking a pair of elements $(a, b) \in A \times B$ and chasing them around both ways, using the fact that $(\tilde{f} \times 1_B)(a, b) = (\tilde{f}(a), b)$.

Now, the property just stated of the set C^B and the evaluation function $\mathrm{eval} : C^B \times B \to C$ is one that will make sense in any category having binary products. It says that evaluation is "the universal map into C from a product with B." Precisely:

Definition 6.1. Let the category $\mathbf{C}$ have binary products. An *exponential* of objects B and C consists of an object

$$C^B$$

and an arrow

$$\epsilon : C^B \times B \to C$$

such that, for any object A and arrow

$$f : A \times B \to C$$

there is a unique arrow

$$\tilde{f} : A \to C^B$$

such that

$$\epsilon \circ (\tilde{f} \times 1_B) = f$$

all as in the diagram

$$\begin{tikzcd} C^B & C^B \times B \arrow[r, "\epsilon"] & C \\ A \arrow[u, "\tilde{f}"] & A \times B \arrow[u, "\tilde{f} \times 1_B"] \arrow[ur, "f"'] & \end{tikzcd}$$

Here is some terminology

- $\epsilon : C^B \times B \to C$ is called *evaluation*.
- $\tilde{f} : A \to C^B$ is called the (exponential) *transpose* of f.
- Given any arrow

  $$g : A \to C^B$$

  we write

  $$\bar{g} := \epsilon \circ (g \times 1_B) : A \times B \to C$$

and also call $\bar{g}$ the *transpose* of g. By the uniqueness clause of the definition, we then have

$$\tilde{\bar{g}} = g$$

and for any $f : A \times B \to C$,

$$\bar{\tilde{f}} = f.$$

Briefly, transposition of transposition is the identity.

Thus in sum, the transposition operation

$$(f : A \times B \to C) \;\longmapsto\; (\tilde{f} : A \to C^B)$$

provides an inverse to the induced operation

$$(g : A \to C^B) \;\longmapsto\; (\bar{g} = \epsilon \circ (g \times 1_B) : A \times B \to C),$$

yielding the desired isomorphism,

$$\mathrm{Hom}_{\mathbf{C}}(A \times B, C) \;\cong\; \mathrm{Hom}_{\mathbf{C}}(A, C^B).$$

6.2 Cartesian closed categories

Definition 6.2. A category is called *cartesian closed* if it has all finite products and exponentials.

Example 6.3. We already have **Sets** as one example, but note that also $\mathbf{Sets}_{fin}$ is cartesian closed, since for finite sets M, N, the set of functions N^M has cardinality

$$|N^M| = |N|^{|M|}$$

and so is also finite.

Example 6.4. Recall that the category **Pos** of posets has as arrows $f : P \to Q$ the monotone functions, $p \leq p'$ implies $fp \leq fp'$. Given posets P and Q, the poset $P \times Q$ has pairs (p, q) as elements, and is partially ordered by

$$(p, q) \leq (p', q') \quad \text{iff} \quad p \leq p' \;\text{ and }\; q \leq q'.$$

Thus the evident projections

$$P \xleftarrow[\pi_1]{} P \times Q \xrightarrow[\pi_2]{} Q$$

are monotone, as is the pairing

$$\langle f, g\rangle : X \to P \times Q$$

if $f : X \to P$ and $g : X \to Q$ are monotone.

For the exponential Q^P, we take the set of monotone functions,

$$Q^P = \{f : P \to Q \mid f \text{ monotone }\}$$

ordered *pointwise*, that is,

$$f \leq g \quad \text{iff} \quad fp \leq gp \text{ for all } p \in P.$$

The evaluation

$$\epsilon : Q^P \times P \to Q$$

and transposition

$$\tilde{f} : X \to Q^P$$

of a given arrow

$$f : X \times P \to Q$$

are the usual ones of the underlying functions. Thus we need only show that these are monotone.

To that end, given $(f, p) \leq (f', p')$ in $Q^P \times P$ we have

$$\begin{aligned} \epsilon(f, p) &= f(p) \\ &\leq f(p') \\ &\leq f'(p') \\ &= \epsilon(f', p') \end{aligned}$$

so ϵ is monotone. Now take $f : X \times P \to Q$ monotone and let $x \leq x'$. We need to show

$$\tilde{f}(x) \leq \tilde{f}(x') \quad \text{in } Q^P$$

which means

$$\tilde{f}(x)(p) \leq \tilde{f}(x')(p) \quad \text{for all } p \in P.$$

But $\tilde{f}(x)(p) = f(x, p) \leq f(x', p) = \tilde{f}(x')(p)$.

Example 6.5. Now let us consider what happens if we restrict to the category of ωCPOs (see example **??**). Given two ωCPOs P and Q, we will take as an exponential the subset,

$$Q^P = \{f : P \to Q \mid f \text{ monotone and } \omega\text{-continuous}\}.$$

Then take evaluation $\epsilon : Q^P \times P \to Q$ and transposition as before, for functions. Then, since we know that the required equations are satisfied, we just need to check the following:

- Q^P is an ωCPO

- ϵ is ω-continuous
- $\tilde{f}$ is ω-continuous if f is

We leave this as an exercise!

Example 6.6. An example of a somewhat different sort is provided by the category **Graphs** of graphs and their homomorphisms. Recall that a graph G consists of a pair of sets G_e and G_v — the edges and vertices — and a pair of functions,

$$\begin{tikzcd} G_e \arrow[d, shift right, "s_G"'] \arrow[d, shift left, "t_G"] \\ G_v \end{tikzcd}$$

called the source and target maps. A homomorphism of graphs $h : G \to H$ is a mapping of edges to edges and vertices to vertices, preserving sources and targets, i.e. is a pair of maps $h_v : G_v \to H_v$ and $h_e : G_e \to H_e$, making the two obvious squares commute.

$$\begin{tikzcd} G_e \arrow[r, "h_e"] \arrow[d, shift right, "s_G"'] \arrow[d, shift left, "t_G"] & H_e \arrow[d, shift right, "s_H"'] \arrow[d, shift left, "t_H"] \\ G_v \arrow[r, "h_v"] & H_e \end{tikzcd}$$

The product $G \times H$ of two graphs G and H, like the product of categories, has as vertices the pairs (g, h) of vertices $g \in G$ and $h \in H$, and similarly the edges are pairs of edges (u, v) with u an edge in G and v and edge in H. The source and target operations are then "pointwise": $s(u, v) = (s(u), s(v))$, etc.

$$\begin{tikzcd} G_e \times H_e \arrow[d, shift right, "s_G \times s_H"'] \arrow[d, shift left, "t_G \times t_H"] \\ G_v \times H_v \end{tikzcd}$$

Now, the exponential graph H^G has as vertices the (arbitrary!) maps of vertices $\varphi : G_v \to H_v$. An edge θ from φ to another vertex $\psi : G_v \to H_v$ is a family of edges (θ_e) in H, one for each edge $e \in G$, such that $s(\theta_e) = \varphi(s(e))$ and $t(\theta_e) = \psi(t(e))$. In other words, θ is a map $\theta : G_e \to H_e$ making hte following commute.

$$\begin{tikzcd}
G_v \arrow[d, "\varphi"'] & G_e \arrow[l, "s"'] \arrow[r, "t"] \arrow[d, "\theta"] & G_v \arrow[d, "\psi"] \\
H_v & H_e \arrow[l, "s"] \arrow[r, "t"'] & H_v
\end{tikzcd}$$

Imagining G as a certain configuration of edges and vertices, and the maps φ and ψ as two different "pictures" or "images" of the vertices of G in H, the edge $\theta : \varphi \to \psi$ appears as a family of edges in H, labeled by the edges of G, each connecting the source vertex in φ to the corresponding target one in ψ. (The reader should draw a diagram at this point.) The evaluation homomorphism $\epsilon : H^G \times G \to H$ takes a vertex (φ, g) to the vertex $\varphi(g)$, and an edge (θ, e) to the edge θ_e. The transpose of a graph homomorphism $f : F \times G \to H$ is the homomorphism $\tilde{f} : F \to H^G$ taking a vertex $a \in F$ to the mapping on vertices $f(a, -) : G_v \to H_v$, and an edge $c : a \to b$ in F to the mapping of edges $f(c, -) : G_e \to H_e$.

We leave the verification of this cartesian closed structure as an exercise for the reader.

Next, we derive some of the basic facts about exponentials and cartesian closed categories. First, let us ask, what is the transpose of evaluation?

$$\epsilon : B^A \times A \to B$$

It must be an arrow $\tilde{\epsilon} : B^A \to B^A$ such that

$$\epsilon(\tilde{\epsilon} \times 1_A) = \epsilon$$

that is, making the following diagram commute:

$$\begin{tikzcd}
B^A \times A \arrow[r, "\epsilon"] & B \\
B^A \times A \arrow[u, "\tilde{\epsilon} \times 1_A"] \arrow[ur, "\epsilon"'] &
\end{tikzcd}$$

Since $1_{B^A} \times 1_A = 1_{(B^A \times A)}$ clearly has this property, we must have

$$\tilde{\epsilon} = 1_{B^A}$$

and so we also know that $\epsilon = \overline{(1_{B^A})}$.

Now let us show that the operation $X \mapsto X^A$ on a CCC is *functorial*.

Proposition 6.7. *In any cartesian closed category* $\mathbf{C}$*, exponentiation by a fixed object* A *is a functor,*

$$(-)^A : \mathbf{C} \to \mathbf{C}.$$

Toward the proof, consider first the case of sets. Given some function

$$\beta : B \to C$$

we put

$$\beta^A : B^A \to C^A$$

defined by

$$f \mapsto \beta \circ f.$$

That is,

$$\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, "\beta \circ f = \beta^A(f)"] & \\ B \arrow[r, "\beta"'] & C \end{tikzcd}$$

This assignment is functorial, because: for any $\alpha : C \to D$

$$\begin{aligned} (\alpha \circ \beta)^A(f) &= \alpha \circ \beta \circ f \\ &= \alpha \circ \beta^A(f) \\ &= \alpha^A \circ \beta^A(f). \end{aligned}$$

Whence $(\alpha \circ \beta)^A = \alpha^A \circ \beta^A$. Also

$$\begin{aligned} (1_B)^A(f) &= 1_B \circ f \\ &= f \\ &= 1_{B^A}(f). \end{aligned}$$

So $(1_B)^A = 1_{B^A}$. Thus $(-)^A$ is indeed a functor; of course, it is just the representable functor $\mathrm{Hom}(A, -)$ that we have already considered.

In a general CCC then, given $\beta : B \to C$, we define

$$\beta^A : B^A \to C^A$$

by

$$\beta^A := \widetilde{(\beta \circ \epsilon)}.$$

That is, we take the transpose of the composite

$$B^A \times A \xrightarrow{\epsilon} B \xrightarrow{\beta} C$$

giving

$$\beta^A : B^A \to C^A.$$

It is easier to see in the form

$$\begin{tikzcd} C^A & C^A \times A \arrow[r, "\epsilon"] & C \\ B^A \arrow[u, "\beta^A"] & B^A \times A \arrow[u, "\beta^A \times 1_A"'] \arrow[r, "\epsilon"'] & B \arrow[u, "\beta"'] \end{tikzcd}$$

Now, clearly,

$$(1_B)^A = 1_{B^A} : B^A \to B^A$$

by examining

$$\begin{tikzcd} B^A \times A \arrow[r, "\epsilon"] & B \\ B^A \times A \arrow[u, "1_{(B^A \times A)} = 1_{B^A} \times 1_A"] \arrow[r, "\epsilon"'] & B \arrow[u, "1_B"'] \end{tikzcd}$$

Quite similarly, given

$$B \xrightarrow{\beta} C \xrightarrow{\gamma} D$$

we have

$$\gamma^A \circ \beta^A = (\gamma \circ \beta)^A.$$

This follows from considering the commutative diagram

$$\begin{tikzcd}
D^A \times A \arrow[r, "\epsilon"] & D \\
C^A \times A \arrow[u, "\gamma^A \times 1_A"] \arrow[r, "\epsilon"'] & C \arrow[u, "\gamma"'] \\
B^A \times A \arrow[u, "\beta^A \times 1_A"] \arrow[r, "\epsilon"'] & B \arrow[u, "\beta"']
\end{tikzcd}$$

We use the fact that

$$(\gamma^A \times 1_A) \circ (\beta^A \times 1_A) = ((\gamma^A \circ \beta^A) \times 1_A).$$

The result follows by the uniqueness of transposes.

There is also another distinguished "universal" arrow; rather than transposing $1_{B^A} : B^A \to B^A$, we can transpose the identity $1_{A\times B} : A \times B \to A \times B$, to get

$$\tilde{1}_{A\times B} : A \to (A \times B)^B.$$

In **Sets** it has the values $\tilde{1}_{A\times B}(a)(b) = (a, b)$. Let us denote this map by $\eta = \tilde{1}_{A\times B}$, so that

$$\eta(a)(b) = (a, b).$$

The map η lets us compute $\tilde{f}$ from the functor $-^A$. Indeed, given $f : Z \times A \to B$ take

$$f^A : (Z \times A)^A \to B^A$$

and precompose with $\eta : Z \to (Z \times A)^A$, as indicated in

$$\begin{tikzcd}
(Z \times A)^A \arrow[r, "f^A"] & B^A \\
Z \arrow[u, "\eta"] \arrow[ur, "\tilde{f}"'] &
\end{tikzcd}$$

This gives the useful equation

$$\tilde{f} = f^A \circ \eta$$

which the reader should prove.

6.3 Heyting algebras

Any Boolean algebra B, regarded as a poset category, has finite products 1 and $a \wedge b$. We can also define the exponential in B by

$$b^a = (\neg a \vee b)$$

which we will also write $a \Rightarrow b$. The evaluation arrow is

$$(a \Rightarrow b) \wedge a \leq b.$$

This always holds since

$$(\neg a \vee b) \wedge a = (\neg a \wedge a) \vee (b \wedge a) = 0 \vee (b \wedge a) = b \wedge a \leq b.$$

To show that $a \Rightarrow b$ is indeed an exponential in B, we just need to verify that if $a \wedge b \leq c$ then $a \leq b \Rightarrow c$, that is, transposition. But if $a \wedge b \leq c$, then

$$\neg b \vee (a \wedge b) \leq \neg b \vee c = b \Rightarrow c.$$

But we also have

$$a \leq \neg b \vee a \leq (\neg b \vee a) \wedge (\neg b \vee b) = \neg b \vee (a \wedge b).$$

This example suggests generalizing the notion of a Boolean algebra to that of a cartesian closed poset. Indeed, consider first the following stronger notion.

Definition 6.8. A *Heyting algebra* is a poset with

1. finite meets: 1 and $p \wedge q$,
2. finite joins: 0 and $p \vee q$,
3. exponentials: for each a, b, an element $a \Rightarrow b$ such that

$$a \wedge b \leq c \quad \text{iff} \quad a \leq b \Rightarrow c.$$

The stated condition on exponentials $a \Rightarrow b$ is equivalent to the UMP in the case of posets. Indeed, given the condition, the transpose of $a \wedge b \leq c$ is $a \leq b \Rightarrow c$ and the evaluation $(a \Rightarrow b) \wedge a \leq b$ follows immediately from $a \Rightarrow b \leq a \Rightarrow b$ (the converse is just as simple).

First, observe that every Heyting algebra is a *distributive lattice*, i.e. for any a, b, c one has:

$$(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$$

Indeed, we have:

$$\begin{aligned}
(a \vee b) \wedge c \leq z \ &\text{iff}\ a \vee b \leq c \Rightarrow z \\
&\text{iff}\ a \leq c \Rightarrow z \ \text{ and } \ b \leq c \Rightarrow z \\
&\text{iff}\ a \wedge c \leq z \ \text{ and } \ b \wedge c \leq z \\
&\text{iff}\ (a \wedge c) \vee (b \wedge c) \leq z.
\end{aligned}$$

Now pick $z = (a \vee b) \wedge c$ and read the equivalences downwards to get one direction, then do the same with $z = (a \wedge c) \vee (b \wedge c)$ and reading the equivalences upwards to get the other direction.

Remark 6.9. The foregoing distributivity is actually a special case of the more general fact that in a cartesian closed category with coproducts, the products necessarily distribute over the coproducts,

$$(A + B) \times C \cong (A \times C) + (B \times C).$$

Although we could prove this now directly, a much more elegant proof (generalizing the one above for the poset case) will be available to us once we have access to the Yoneda Lemma. For this reason, we defer the proof of distributivity to **??** below.

One may well wonder whether all distributive lattices are Heyting algebras. The answer is in general, no; but certain ones always are.

Definition 6.10. A poset is *(co)complete* if it is so as a category, thus if it has all set-indexed meets $\bigwedge_{i \in I} a_i$ (resp. joins $\bigvee_{i \in I} a_i$). For posets, completeness and cocompleteness are equivalent (exercise!). A lattice, Heyting algebra, Boolean algebra, etc. is called *complete* if it is so as a poset.

Proposition 6.11. *A complete lattice is a Heyting algebra iff it satisfies the infinite distributive law*

$$a \wedge \left(\bigvee_i b_i \right) = \bigvee_i (a \wedge b_i).$$

Proof. One shows that Heyting algebra implies distributivity just as in the finite case. To show that the infinite distributive law implies Heyting algebra, set

$$a \Rightarrow b = \bigvee_{x \wedge a \leq b} x.$$

Then, if

$$y \wedge a \leq b$$

then $y \leq \bigvee_{x \wedge a \leq b} x = a \Rightarrow b$. And conversely, if $y \leq a \Rightarrow b$ then $y \wedge a \leq (\bigvee_{x \wedge a \leq b} x) \wedge a = \bigvee_{x \wedge a \leq b} (x \wedge a) \leq \bigvee b = b$. □

Example 6.12. For any set A, the powerset $P(A)$ is a complete Heyting algebra with unions and intersections as joins and meets, since it satisfies the infinite distributive law. More generally, the lattice of open sets of a topological space is also a Heyting algebra, since the open sets are closed under finite intersections and arbitrary unions.

Of course, every Boolean algebra is a Heyting algebra with $a \Rightarrow b = \neg a \vee b$, as we already showed. But in general, a Heyting algebra is not Boolean. Indeed, we can define a proposed negation by,

$$\neg a = a \Rightarrow 0$$

as must be the case, since in a Boolean algebra $\neg a = \neg a \vee 0 = a \Rightarrow 0$. Then $a \leq \neg\neg a$ since $a \wedge (a \Rightarrow 0) \leq 0$. But, conversely, $\neg\neg a \leq a$ need not hold in a Heyting algebra. Indeed, in a topological space X, the negation $\neg U$ of an open subset U is the *interior* of the complement $X - U$. Thus, for example, in the real interval $[0, 1]$ we have $\neg\neg(0, 1) = [0, 1]$.

Moreover, the law,

$$1 \leq a \vee \neg a$$

also need not hold in general. In fact, the concept of a Heyting algebra is the algebraic equivalent of the *intuitionistic* propositional calculus, in the same sense that Boolean algebras are an algebraic formulation of the *classical* propositional calculus.

6.4 Propositional calculus

In order to make the connection between Heyting algebras and propositional calculus more rigorous, let us first give a specific system of rules for the intuitionistic propositional calculus (IPC). This we do in terms of entailments $p \vdash q$ between formulas p and q:

1. $\vdash$ is reflexive and transitive
2. $p \vdash \top$
3. $\bot \vdash p$
4. $p \vdash q$ and $p \vdash r$ iff $p \vdash q \wedge r$
5. $p \vdash r$ and $q \vdash r$ iff $p \vee q \vdash r$
6. $p \wedge q \vdash r$ iff $p \vdash q \Rightarrow r$

This is a complete system for IPC, equivalent to the more standard presentations the reader may have seen. To compare with one perhaps more familiar presentation, note first that we have an "evaluation" entailment by reflexivity and (6):

$$\frac{p \Rightarrow q \vdash p \Rightarrow q}{(p \Rightarrow q) \wedge p \vdash q}$$

We therefore have the rule of "modus ponens" by (4) and transitivity:

$$\begin{aligned}
&\top \vdash p \Rightarrow q \quad \text{and} \quad \top \vdash p \\
&\top \vdash (p \Rightarrow q) \wedge p \\
&\top \vdash q
\end{aligned}$$

Moreover, by (4) there are "projections":

$$\begin{aligned}
&p \wedge q \vdash p \wedge q \\
&p \wedge q \vdash p \quad (\text{resp. } q)
\end{aligned}$$

from which it follows that $p \dashv\vdash \top \wedge p$. Thus, we get one of the usual axioms for products:

$$\begin{aligned}
p \wedge q &\vdash p \\
\top \wedge (p \wedge q) &\vdash p \\
\top &\vdash (p \wedge q) \Rightarrow p
\end{aligned}$$

Now let us derive the usual axioms for $\Rightarrow$, namely:

1. $p \Rightarrow p$,
2. $p \Rightarrow (q \Rightarrow p)$,
3. $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$.

The first two are almost immediate:

$$\begin{aligned}
p &\vdash p \\
\top \wedge p &\vdash p \\
\top &\vdash p \Rightarrow p
\end{aligned}$$

$$\begin{aligned}
p \wedge q &\vdash p \\
p &\vdash q \Rightarrow p \\
\top \wedge p &\vdash (q \Rightarrow p) \\
\top &\vdash p \Rightarrow (q \Rightarrow p)
\end{aligned}$$

For the third one, we shall use the fact that $\Rightarrow$ distributes over $\wedge$ on the right:

$$a \Rightarrow (b \wedge c) \dashv\vdash (a \Rightarrow b) \wedge (a \Rightarrow c)$$

This is a special case of the exercise:

$$(B \times C)^A \;\cong\; B^A \times C^A$$

We also use the following simple fact, which will be recognized as a special case of proposition 6.7:

$$a \vdash b \quad \text{implies} \quad p \Rightarrow a \vdash p \Rightarrow b \tag{6.2}$$

Then we have,

$$\begin{aligned}
(q \Rightarrow r) \wedge q &\vdash r \\
p \Rightarrow ((q \Rightarrow r) \wedge q) &\vdash p \Rightarrow r \\
(p \Rightarrow (q \Rightarrow r)) \wedge (p \Rightarrow q) &\vdash p \Rightarrow r \qquad \text{by (6.3)} \\
(p \Rightarrow (q \Rightarrow r)) &\vdash (p \Rightarrow q) \Rightarrow (p \Rightarrow r) \\
\top &\vdash (p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \wedge (p \Rightarrow r)).
\end{aligned}$$

The "positive" fragment of IPC, involving only the logical operations

$$\top, \ \wedge, \ \Rightarrow$$

corresponds to the notion of a cartesian closed poset. We then add $\bot$ and disjunction $p \vee q$ on the logical side and finite joins on the algebraic side to arrive at a correspondence between IPC and Heyting algebras. The exact correspondence is given by mutually inverse constructions between Heyting algebras and intuitionistic propositional calculi. We briefly indicate one direction of this correspondence, leaving the other one to the reader's ingenuity.

Given any intuitionistic propositional calculus $\mathcal{L}$, consisting of propositional formulas $p, q, r, \ldots$ over some set of variables $x, y, z, \ldots$ together with the rules of inference stated above, and perhaps some distinguished formulas $a, b, c, \ldots$ as axioms, one constructs from $\mathcal{L}$ a Heyting algebra $\mathrm{HA}(\mathcal{L})$, called the *Lindenbaum-Tarski algebra*, consisting of equivalence classes $[p]$ of formulas p, where:

$$[p] = [q] \quad \text{iff} \quad p \dashv\vdash q \tag{6.3}$$

The ordering in $\mathrm{HA}(\mathcal{L})$ is given by:

$$[p] \leq [q] \quad \text{iff} \quad p \vdash q \tag{6.4}$$

This is clearly well defined on equivalence classes, in the sense that if $p \vdash q$ and $[p] = [p']$ then $p' \vdash q$, and similarly for q. The operations in $\mathrm{HA}(\mathcal{L})$ are then induced in the expected way by the logical operations:

$$\begin{aligned}
1 &= [\top] \\
0 &= [\bot] \\
[p] \wedge [q] &= [p \wedge q] \\
[p] \vee [q] &= [p \vee q] \\
[p] \Rightarrow [q] &= [p \Rightarrow q]
\end{aligned}$$

Again, these operations are easily seen to be well defined on equivalence classes, and they satisfy the laws for a Heyting algebra because the logical rules evidently imply them.

Lemma 6.13. *Observe that, by* (6.3)*, the Heyting algebra* $\mathrm{HA}(\mathcal{L})$ *has the property that a formula* p *is provable* $\top \vdash p$ *if and only if* $[p] = 1$.

Now define an *interpretation* M of $\mathcal{L}$ in a Heyting algebra H to be an assignment of the basic propositional variables $x, y, z, \ldots$ to elements of H, which we shall write as $[\![x]\!], [\![y]\!], [\![z]\!], \ldots$. An interpretation then extends to all formulas by recursion in the evident way, i.e. $[\![p \wedge q]\!] = [\![p]\!] \wedge [\![q]\!]$, etc. An interpretation is called a *model* of $\mathcal{L}$ if for every theorem $\top \vdash p$, one has $[\![p]\!] = 1$. Observe that there is a canonical interpretation of $\mathcal{L}$ in $\mathrm{HA}(\mathcal{L})$ given by $[\![x]\!] = [x]$. One shows easily by induction that, for any formula p, moreover, $[\![p]\!] = [p]$. Now Lemma 6.13 tells us that this interpretation is in fact a model of $\mathcal{L}$ and that, moreover, it is "generic," in the sense that it validates *only* the provable formulas. We therefore have the following logical *completeness theorem for IPC.*

Proposition 6.14. *The intuitionistic propositional calculus is complete with respect to models in Heyting algebras.*

Proof. Suppose a formula p is true in all models in all Heyting algebras. Then in particular, it is so in $\mathrm{HA}(\mathcal{L})$. Thus $1 = [\![p]\!] = [p]$ in $\mathrm{HA}(\mathcal{L})$, and so $\top \vdash p$. $\square$

In sum, then, a particular instance $\mathcal{L}$ of intuitionistic propositional calculus can be regarded as a way of specifying (and reasoning about) a particular Heyting algebra $\mathrm{HA}(\mathcal{L})$. Indeed, it is essentially a presentation by generators and relations, in just the way that we have already seen for other algebraic objects like monoids. The Heyting algebra $\mathrm{HA}(\mathcal{L})$ even has a UMP with respect to $\mathcal{L}$ that is entirely analogous to the UMP of a finitely presented monoid given by generators and relations. Specifically, if, for instance, $\mathcal{L}$ is generated by the two elements x, y subject to the single "axiom" $x \vee y \Rightarrow x \wedge y$, then in $\mathrm{HA}(\mathcal{L})$ the elements $[x]$ and $[y]$ satisfy $[x] \vee [y] \leq [x] \wedge [y]$ (which is of course equivalent to $([x] \vee [y] \Rightarrow [x] \wedge [y]) = 1$), and given any Heyting algebra A with two elements a and b satisfying $a \vee b \leq a \wedge b$, there is a unique Heyting homomorphism $h : \mathrm{HA}(\mathcal{L}) \to A$ with $h([x]) = a$ and $h([y]) = b$. In this sense, the Lindenbaum-Tarski Heyting algebra $\mathrm{HA}(\mathcal{L})$, being finitely presented by the generators and axioms of $\mathcal{L}$, can be said to contain a "universal model" of the theory determined by $\mathcal{L}$.

6.5 Equational definition of CCC

The following description of CCCs in terms of operations and equations on a category is often useful. The proof is entirely routine and left to the reader.

Proposition 6.15. *A category* $\mathbf{C}$ *is a CCC iff it has the following structure:*

- *A distinguished object* 1*, and for each object* C *there is given an arrow*

  $$!_C : C \to 1$$

  such that for each arrow $f : C \to 1$*,*

  $$f = !_C.$$

- *For each pair of objects* A, B*, there is given an object* $A \times B$ *and arrows,*

  $$p_1 : A \times B \to A \quad \textit{and} \quad p_2 : A \times B \to B$$

  and for each pair of arrows $f : Z \to A$ *and* $g : Z \to B$*, there is given an arrow,*

  $$\langle f, g \rangle : Z \to A \times B$$

  such that:

  $$\begin{aligned} p_1\langle f, g\rangle &= f \\ p_2\langle f, g\rangle &= g \\ \langle p_1 h, p_2 h\rangle &= h \quad \textit{for all } h : Z \to A \times B. \end{aligned}$$

- *For each pair of objects* A, B*, there is given an object* B^A *and an arrow,*

  $$\epsilon : B^A \times A \to B$$

  and for each arrow $f : Z \times A \to B$ *there is given an arrow*

  $$\tilde{f} : Z \to B^A$$

  such that:

  $$\epsilon \circ (\tilde{f} \times 1_A) = f$$

  and

  $$(\epsilon \circ \widetilde{(g \times 1_A)}) = g$$

  for all $g : Z \to B^A$*. Here, and generally, for any* $a : X \to A$ *and* $b : Y \to B$*, we write:*

  $$a \times b = \langle a \circ p_1, b \circ p_2\rangle : X \times Y \to A \times B.$$

It is sometimes easier to check these equational conditions than to verify the corresponding universal mapping properties. The next section provides an example of this sort.

6.6 λ-calculus

We have seen that the notions of a cartesian closed poset with finite joins (i.e. a Heyting algebra) and intuitionistic propositional calculus are essentially the same,

$$\text{HA} \sim \text{IPC}.$$

These are two different ways of describing one and the same structure; whereby, to be sure, the logical description contains some superfluous data in the choice of a particular presentation.

We now want to consider another, very similar, correspondence between systems of logic and categories, involving more general CCC's. Indeed, the foregoing correspondence was the poset case of the following general one between CCCs and λ-calculus,

$$\text{CCC} \ \sim \ \lambda\text{-calculus}.$$

These notions are also essentially equivalent, in a sense that we will now sketch (a more detailed treatment can be found in the book by Lambek and Scott). They are two different ways of representing the same idea, namely that of a collection of objects and functions, with operations of *pairing*, *projection*, *application*, and *transposition* (or "*currying*").

First, recall the notion of a (typed) λ-calculus from Chapter 2. It consists of:

- Types: $A \times B,\ A \to B, \dots$ (and some basic types)
- Terms: $x, y, z, \dots : A$ (variables for each type A)
  $a : A,\ b : B, \dots$ (possibly some typed constants)

$$\begin{array}{rl}
\langle a, b\rangle : A \times B & (a : A,\ b : B) \\
\mathrm{fst}(c) : A & (c : A \times B) \\
\mathrm{snd}(c) : B & (c : A \times B) \\
ca : B & (c : A \to B,\ a : A) \\
\lambda x.b : A \to B & (x : A,\ b : B)
\end{array}$$

- Equations, including at least all instances of the following:

$$\begin{aligned}
\mathrm{fst}(\langle a, b\rangle) &= a \\
\mathrm{snd}(\langle a, b\rangle) &= b \\
\langle \mathrm{fst}(c), \mathrm{snd}(c)\rangle &= c \\
(\lambda x.b)a &= b[a/x] \\
\lambda x.cx &= c \quad (\text{no } x \text{ in } c)
\end{aligned}$$

Given a particular such λ-calculus $\mathcal{L}$, the associated *category of types* $\mathbf{C}(\mathcal{L})$ was then defined as follows:

- objects: the types,
- arrows $A \to B$: equivalence classes of closed terms $[c] : A \to B$, identified according to (renaming of bound variables and),

$$[a] = [b] \quad \text{iff } \mathcal{L} \vdash a = b \tag{6.5}$$

- identities: $1_A = [\lambda x.x]$ (where $x : A$),
- composition: $[c] \circ [b] = [\lambda x.c(bx)]$.

We have already seen that this is a well-defined category, and that it has binary products. It is a simple matter to add a terminal object. Now let us use the equational characterization of CCCs to show that it is cartesian closed. Given any objects A, B, we set $B^A = A \to B$, and as the evaluation arrow we take (the equivalence class of),

$$\epsilon = \lambda z.\mathrm{fst}(z)\mathrm{snd}(z) : B^A \times A \to B \quad (z : Z).$$

Then for any arrow $f : Z \times A \to B$, we take as the transpose,

$$\tilde{f} = \lambda z \lambda x.f\langle z, x\rangle : Z \to B^A \quad (z : Z,\ x : A).$$

It is now a straightforward λ-calculus calculation to verify the two required equations, namely,

$$\epsilon \circ (\tilde{f} \times 1_A) = f,$$
$$(\epsilon \circ \widetilde{(g \times 1_A)}) = g.$$

In detail, for the first one recall that

$$\alpha \times \beta = \lambda w.\langle \alpha\mathrm{fst}(w), \beta\mathrm{snd}(w)\rangle.$$

So we have

$$\begin{aligned}
\epsilon \circ (\tilde{f} \times 1_A) &= (\lambda z.\mathrm{fst}(z)\mathrm{snd}(z)) \circ [(\lambda y \lambda x.f\langle y, x\rangle) \times \lambda u.u] \\
&= \lambda v.(\lambda z.\mathrm{fst}(z)\mathrm{snd}(z))[(\lambda y \lambda x.f\langle y, x\rangle) \times \lambda u.u]v \\
&= \lambda v.(\lambda z.\mathrm{fst}(z)\mathrm{snd}(z))[\lambda w.\langle(\lambda y \lambda x.f\langle y, x\rangle)\mathrm{fst}(w), (\lambda u.u)\mathrm{snd}(w)\rangle]v \\
&= \lambda v.(\lambda z.\mathrm{fst}(z)\mathrm{snd}(z))[\lambda w.\langle(\lambda x.f\langle \mathrm{fst}(w), x\rangle), \mathrm{snd}(w)\rangle]v \\
&= \lambda v.(\lambda z.\mathrm{fst}(z)\mathrm{snd}(z))[\langle(\lambda x.f\langle \mathrm{fst}(v), x\rangle), \mathrm{snd}(v)\rangle] \\
&= \lambda v.(\lambda x.f\langle \mathrm{fst}(v), x\rangle)\mathrm{snd}(v) \\
&= \lambda v.f\langle \mathrm{fst}(v), \mathrm{snd}(v)\rangle \\
&= \lambda v.fv \\
&= f.
\end{aligned}$$

The second equation is proved similarly.

Let us call a set of basic types and terms, together with a set of equations between terms, a *theory* in the λ-calculus. Given such a theory $\mathcal{L}$, the cartesian closed category $\mathbf{C}(\mathcal{L})$ built from the λ-calculus over $\mathcal{L}$ is the CCC presented by the generators and relations stated by $\mathcal{L}$. Just as in the poset case of IPC and Heyting algebras, there is a logical completeness theorem that follows from this fact. To state it, we require the notion of a model of a theory $\mathcal{L}$ in the λ-calculus in an arbitrary cartesian closed category $\mathbf{C}$. We give only a brief sketch to give the reader the general idea:

Definition 6.16. A *model* of $\mathcal{L}$ in $\mathbf{C}$ is an assignment of the types and terms of $\mathcal{L}$ to objects and arrows of $\mathbf{C}$:

$$\begin{array}{rcl} X \text{ basic type} & \rightsquigarrow & [\![X]\!] \text{ object} \\ b : A \to B \text{ basic term} & \rightsquigarrow & [\![b]\!] : [\![A]\!] \to [\![B]\!] \text{ arrow} \end{array}$$

This assignment is then extended to all types and terms in such a way that the λ-calculus operations are taken to the corresponding CCC ones:

$$[\![A \times B]\!] = [\![A]\!] \times [\![B]\!]$$
$$[\![\langle f, g\rangle]\!] = \langle [\![f]\!], [\![g]\!]\rangle$$
$$\text{etc.}$$

Finally, it is required that all the equations of $\mathcal{L}$ are satisfied, in the sense that:

$$\mathcal{L} \vdash [a] = [b] : A \to B \quad \text{implies} \quad [\![a]\!] = [\![b]\!] : [\![A]\!] \to [\![B]\!] \tag{6.6}$$

This is what is sometimes called "denotational semantics" for the λ-calculus. It is essentially the conventional, set-theoretic semantics for first-order logic, but extended to higher types, restricted to equational theories, and generalized to CCCs.

For example, let $\mathcal{L}$ be the theory with one basic type X, two basic terms,

$$u : X$$
$$m : X \times X \to X$$

and the usual equations for associativity and units,

$$m\langle u, x\rangle = x$$
$$m\langle x, u\rangle = x$$
$$m\langle x, m\langle y, z\rangle\rangle = m\langle m\langle x, y\rangle, z\rangle.$$

Thus $\mathcal{L}$ is just the usual equational theory of monoids. Then a model of $\mathcal{L}$ in a cartesian closed category $\mathbf{C}$ is nothing but a monoid in $\mathbf{C}$, that is, an object $M = [\![X]\!]$ equipped with a distinguished point

$$[\![u]\!] : 1 \to M$$

and a binary operation

$$[\![m]\!] : M \times M \to M$$

satisfying the unit and associativity laws.

Note that by (6.5) and (6.6), there is a model of $\mathcal{L}$ in $\mathbf{C}(\mathcal{L})$ with the property that $[\![a]\!] = [\![b]\!] : X \to Y$ if and only if $a = b$ is provable in $\mathcal{L}$. In this way, one can prove the following *CCC completeness theorem for λ-calculus.*

Proposition 6.17. *For any theory $\mathcal{L}$ in the λ-calculus, one has the following.*

1. *For any terms a, b, $\mathcal{L} \vdash a = b$ iff for all models M in CCCs, $[\![a]\!]_M = [\![b]\!]_M$.*
2. *Moreover, for any type A, there is a closed $t : A$ iff for all models M in CCCs, there is an arrow $1 \to [\![A]\!]_M$.*

This proposition says that the λ-calculus is *deductively sound and complete* for models in CCCs. It is worth emphasizing that completeness is not true if one restricts attention to models in the single category **Sets**; indeed there are many examples of theories in λ-calculus in which equations holding for all models in **Sets** are still not provable (see the exercises for an example).

Soundness (i.e. the "only if" direction of the above statements) follows from the following UMP of the cartesian closed category $\mathbf{C}(\mathcal{L})$, analogous to the one for any algebra presented by generators and relations. Given any model M of $\mathcal{L}$ in any cartesian closed category $\mathbf{C}$, there is a unique functor,

$$[\![-]\!]_M : \mathbf{C}(\mathcal{L}) \to \mathbf{C}$$

preserving the CCC structure, given by,

$$[\![X]\!]_M = M$$

for the basic type X, and similarly for the other basic types and terms of $\mathcal{L}$. In this precise sense, the theory $\mathcal{L}$ is a presentation of the cartesian closed category $\mathbf{C}(\mathcal{L})$ by generators and relations.

Finally, let us note that the notions of λ-calculus and CCC are essentially "equivalent," in the sense that any cartesian closed category $\mathbf{C}$ also gives rise to a λ-calculus $\mathcal{L}(\mathbf{C})$, and this construction is essentially inverse to the one just sketched.

Briefly, given $\mathbf{C}$, we define $\mathcal{L}(\mathbf{C})$ by:

- basic types: the objects of $\mathbf{C}$
- basic terms: $a : A \to B$ for each $a : A \to B$ in $\mathbf{C}$

- equations: many equations identifying the λ-calculus operations with the corresponding category and CCC structure on $\mathbf{C}$, for example:

$$\begin{aligned}\lambda x.\mathrm{fst}(x) &= p_1\\ \lambda x.\mathrm{snd}(x) &= p_2\\ \lambda y.f(x,y) &= \tilde{f}(x)\\ g(f(x)) &= (g\circ f)(x)\\ \lambda y.y &= 1_A\end{aligned}$$

This suffices to ensure that there is an isomorphism of categories,

$$\mathbf{C}(\mathcal{L}(\mathbf{C})) \cong \mathbf{C}.$$

Moreover, the theories $\mathcal{L}$ and $\mathcal{L}(\mathbf{C}(\mathcal{L}))$ will also be "equivalent" in a suitable sense, involving the kinds of considerations typical of comparing different presentations of algebras. We refer the reader to the excellent book by Lambek and Scott (1986) for further details.

6.7 Variable sets

We conclude with a special kind of CCC related to so-called "Kripke models" of logic, namely categories of *variable sets*. These categories provide specific examples of the "algebraic" semantics of IPC and λ-calculus just given.

6.7.1 IPC

Let us begin by very briefly reviewing the notion of a Kripke model of IPC from our algebraic point of view; we'll focus in the positive fragment involving only $\top, p\wedge q, p \Rightarrow q$ and variables.

A *Kripke model* of this language $\mathcal{L}$ consists of a poset I of "possible worlds," which we will write $i \leq j$, together with a relation between worlds i and propositions p,

$$i \Vdash p,$$

read "p holds at i". This relation is assumed to satisfy the conditions:

(1) $i \Vdash p$ and $i \leq j$ implies $j \Vdash p$

(2) $i \Vdash \top$

(3) $i \Vdash p \wedge q$ iff $i \Vdash p$ and $i \Vdash q$

(4) $i \Vdash p \Rightarrow q$ iff $j \Vdash p$ implies $j \Vdash q$ for all $j \geq i$.

One then sets:

$$I \Vdash p \quad \text{iff} \quad i \Vdash p \quad \text{for all } i \in I.$$

And finally, we have the well-known,

Theorem 6.18 (Kripke completeness for IPC). *A propositional formula p is provable from the rules for* IPC *iff it holds in all Kripke models, i.e. iff $I \Vdash p$ for all relations $\Vdash$ over all posets I,*

$$\mathrm{IPC} \vdash p \quad \textit{iff} \quad I \Vdash p \quad \textit{for all } I.$$

Now let us see how to relate this result to our formulation of the semantics of IPC in Heyting algebras. First, the relation $\Vdash \subseteq I \times \mathrm{Prop}(\mathcal{L})$ between worlds I and propositional formulas $\mathrm{Prop}(\mathcal{L})$ can be equivalently formulated as a mapping,

$$[\![-]\!] : \mathrm{Prop}(\mathcal{L}) \longrightarrow \mathbf{2}^I, \tag{6.7}$$

where we write $\mathbf{2}^I = \mathrm{Hom}_{\mathbf{Pos}}(I, \mathbf{2})$ for the exponential poset of monotone maps from I into the poset $\mathbf{2} = \{\bot \leq \top\}$. This poset is a CCC, and indeed a Heyting algebra, the proof of which we leave as an exercise for the reader. The mapping (6.7) is determined by the condition:

$$[\![p]\!](i) = \top \quad \text{iff} \quad i \Vdash p.$$

Now, in terms of the Heyting algebra semantics of IPC developed in section 6.4 above (adapted in the evident way to the current setting without the coproducts $\bot, p \vee q$, and writing HA^- for Heyting algebras without coproducts, i.e. poset CCCs), the poset $\mathrm{HA}^-(\mathcal{L})$ is a quotient of $\mathrm{Prop}(\mathcal{L})$ by the equivalence relation of mutual derivability $p \dashv\vdash q$, which clearly makes it a CCC, and the map (6.7) therefore determines a model (with the same name),

$$[\![-]\!] : \mathrm{HA}^-(\mathcal{L}) \longrightarrow \mathbf{2}^I.$$

Indeed, condition (1) above ensures that $[\![p]\!] : I \to \mathbf{2}$ is monotone, and (2)–(4) ensure that $[\![-]\!]$ is a homomorphism of poset CCCs, i.e. that it is monotone and preserves the CCC structure (exercise!). Thus a Kripke model is just an "algebraic" model in a Heyting algebra of the special form $\mathbf{2}^I$. The Kripke completeness theorem for positive IPC above then follows from our Heyting-valued completeness theorem Proposition 6.14 together with the following, purely algebraic, embedding theorem for poset CCCs.

Proposition 6.19. *For every poset CCC $\mathbf{A}$, there is a poset I and an injective, monotone map,*

$$y : \mathbf{A} \rightarrowtail \mathbf{2}^I,$$

preserving CCC structure.

Proof. We can take $I = \mathbf{A}^{\mathrm{op}}$ and $y(a) : \mathbf{A}^{\mathrm{op}} \to \mathbf{2}$ the "truth-value" of $x \leq a$, i.e. $y(a)$ is determined by:

$$y(a)(x) = \top \quad \text{iff} \quad x \leq a.$$

Clearly, $y(a)$ is monotone and contravariant, while y itself is monotone and covariant. We leave it as an exercise to verify that y is injective and preserves the CCC structure, but note that $\mathbf{2}^{\mathbf{A}^{\mathrm{op}}}$ can be identified with the collection of *lower sets* $S \subseteq \mathbf{A}$ in $\mathbf{A}$, i.e. subsets that are closed downwards: $x \leq y \in S$ implies $x \in S$. Under this identification we then have $y(a) = \downarrow(a) = \{x \mid x \leq a\}$.

A proof will also be given in Chapter 8 as a consequence of the Yoneda Lemma. □

The result can be extended from poset CCCs to Heyting algebras, thus recovering the usual Kripke completeness theorem for full IPC, by the same argument using a more delicate embedding theorem that also preserves the coproducts $\bot$ and $p \vee q$.

6.7.2 λ-calculus

We now want to generalize the foregoing from propositional logic to the λ-calculus, motivated by the insight that the latter is the proof theory of the former (according to the Curry-Howard correspondence). Categorically speaking, we are generalizing from the poset case to the general case of a CCC. According to the "propositions-as-types" conception behind the C-H correspondence, we therefore should replace the poset CCC of idealized propositions $\mathbf{2}$ with the general CCC of idealized types **Sets**. We shall therefore model the λ-calculus in categories of the form $\mathbf{Sets}^I$ for posets I, which can be regarded as comprised of "I-indexed," or "variable sets," as we now indicate.

Given a poset I, an *I-indexed set* is a family of sets $(A_i)_{i \in I}$ together with transition functions $\alpha_{ij} : A_i \to A_j$ for each $i \leq j$, satisfying the compatibility conditions:

- $\alpha_{ik} = \alpha_{jk} \circ \alpha_{ij}$ whenever $i \leq j \leq k$,
- $\alpha_{ii} = 1_{A_i}$ for all i.

In other words, it is simply a functor,

$$A : I \longrightarrow \mathbf{Sets}.$$

We can think of such I-indexed sets as "sets varying in a parameter" from the poset I. For instance, if $I = \mathbb{R}$ thought of as time, then an $\mathbb{R}$-indexed set A may be thought of as a set varying through time: some elements $a, b \in A_t$ may become identified over time (the αs need not be injective), and new elements may appear over time (the αs need not be surjective), but once an element is in the set ($a \in A_t$), it stays in forever ($\alpha_{tt'}(a) \in A_{t'}$). For a more general poset I the variation is parameterized accordingly.

A product of two variable sets A and B can be constructed by taking the pointwise products $(A \times B)(i) = A(i) \times B(i)$ with the evident transition maps,

$$\alpha_{ij} \times \beta_{ij} : A(i) \times B(i) \to A(j) \times B(j) \qquad i \leq j$$

where $\beta_{ij} : B_i \to B_j$ is the transition map for B. This plainly gives an I-indexed set, but to check that it really is a product we need to make $\mathbf{Sets}^I$ into a category and verify the UMP (respectively, the operations and equations of section 6.5). What is a map of I-indexed sets $f : A \to B$? One natural proposal is this: it is an I-indexed family of functions $(f_i : A_i \to B_i)_{i \in I}$ that are compatible with the transition maps, in the sense that whenever $i \leq j$ then the following commutes.

$$\begin{tikzcd} A_i \arrow[r, "f_i"] \arrow[d, "\alpha_{ij}"'] & B_i \arrow[d, "\beta_{ij}"] \\ A_j \arrow[r, "f_j"'] & B_j \end{tikzcd}$$

We can think of this condition as saying that f takes elements $a \in A$ to elements $f(a) \in B$ without regard to when the transition is made, since given $a \in A_i$ it doesn't matter if we first wait until $j \geq i$ and then take $f_j(\alpha_{ij}(a))$, or go right away to $f_i(a)$ and then wait until $\beta_{ij}(f_i(a))$.

Indeed, in the next chapter we shall see that this type of map is exactly what is called a "natural transformation" of the functors A and B. These maps $f : A \to B$ compose in the evident way,

$$(g \circ f)_i = g_i \circ f_i : A_i \longrightarrow B_i$$

to make $Sets^I$ into a category, the *category of I-indexed sets*. It is now an easy exercise to confirm that the specification of the product $A \times B$just given really is a product in the resulting category $\mathbf{Sets}^I$, and the terminal object is obviously the constant index set 1, so $\mathbf{Sets}^I$ has all finite products.

What about exponentials? The first attempt at defining pointwise exponentials,

$$(B^A)_i = B_i^{A_i}$$

fails, because the indexing is covariant in B and *contra*variant in A, as the reader should confirm. The idea that maybe B^A is just the collection of all index maps from A to B also fails, because it is not indexed! The solution is a combintion of these two ideas which generalizes the "Kripke" exponential as follows. For each $i \in I$, let

$$\downarrow(i) \subseteq I$$

be the lower set below i, regarded as a subposet. Then for any $A : I \to \mathbf{Sets}$, let $A|_i$ be the restriction,

$$\begin{tikzcd} I \arrow[r, "A"] & \mathbf{Sets} \\ \downarrow(i) \arrow[u, hook] \arrow[ur, "A|_i"'] & \end{tikzcd}$$

This determines an indexed set over $\downarrow(i)$. Given any $f : A \to B$ and $i \in I$ there is an evident restriction $f|_i : A|_i \to B|_i$ which is defined to be simply $(f|_i)_j = f_j$ for any $j \leq i$. Now we can define:

$$(B^A)_i \;=\; \{f : A|_i \to B|_i \mid f \text{ is } \downarrow(i)\text{-indexed}\}$$

with the transition maps given by:

$$f \mapsto f|_j \qquad j \leq i$$

It is immediate that this determines an I-indexed set B^A. That it is actually the exponential of A and B in $\mathbf{Sets}^I$ will be shown later, as an easy consequence of the Yoneda Lemma. For the record, we therefore have the following (proof deferred).

Proposition 6.20. *For any poset I, the category $Sets^I$ of I-indexed sets and functions is cartesian closed.*

Definition 6.21. A *Kripke model* of a theory $\mathcal{L}$ in the λ-calculus is a model (in the sense of definition 6.16) in a cartesian closed category of the form $\mathbf{Sets}^I$ for a poset I.

For instance, it can be seen that a Kripke model over a poset I of a conventional algebraic theory such as the theory of groups is just an I indexed group, i.e. a functor $I \to \mathbf{Group}$. In particular, if $I = \mathcal{O}(X)^{\mathrm{op}}$ for a topological space X, then this is just what the topologist calls a "presheaf of groups." On the other hand, it also agrees with (or generalizes) the logician's notion of a Kripke model of a first-order language, in that it consists of a varying domain of "individuals" equipped with varying structure.

Finally, in order to generalize the Kripke completeness theorem for IPC to λ-calculus, it clearly suffices to sharpen our general CCC completeness theorem, proposition 6.17, to the special models in CCCs of the form $\mathbf{Sets}^I$ by means of an embedding theorem analogous to proposition 6.19 above. Indeed, one can prove:

Proposition 6.22. *For every CCC* $\mathbf{C}$, *there is a poset I and a functor,*

$$y : \mathbf{C} \rightarrowtail \mathbf{Sets}^I,$$

that is injective on both objects and arrows and preserves CCC structure. Moreover, every map between objects in the image of y is itself in the image of y (y is "full").

The full proof of this result involves methods from topos theory that are beyond the scope of this book. But a significant part of it, to be given below, is entirely analogous to the proof of the poset case, and will again be a consequence of the Yoneda Lemma.

6.8 Exercises

1. Show that for all finite sets M and N,

$$|N^M| = |N|^{|M|},$$

where $|K|$ is the number of elements in the set K, while N^M is the exponential in the category of sets (the set of all functions $f : M \to N$), and n^m is the usual exponentiation operation of arithmetic.

2. Show that for any three objects A, B, C in a cartesian closed category, there are isomorphisms:

   (a) $(A \times B)^C \cong A^C \times B^C$
   (b) $(A^B)^C \cong A^{B \times C}$

3. Determine the exponential transpose $\tilde{\varepsilon}$ of evaluation $\varepsilon : B^A \times A \to B$ (for any objects in any CCC). In **Sets**, determine the transpose $\tilde{1}$ of the identity $1 : A \times B \to A \times B$. Also determine the transpose of $\varepsilon \circ \tau : A \times B^A \to B$, where $\tau : A \times B^A \to B^A \times A$ is the "twist" arrow $\tau = \langle p_2, p_1 \rangle$.

4. Is the category of monoids cartesian closed?

5. Verify the description given in the text of the exponential graph H^G for two graphs G and H. Determine the exponential $\mathbf{2}^G$, where $\mathbf{2}$ is the graph $v_1 \to v_2$ with two vertices and one edge, and G is an arbitrary graph. Determine $\mathbf{2}^G$ explicitely for G the graph pictured below.

$$\begin{tikzcd}
a \arrow[r] & b & c \arrow[l] \\
 & d \arrow[u] &
\end{tikzcd}$$

6. Consider the category of sets equipped with a (binary) relation, $(A, R \subseteq A \times A)$, with maps $f : (A, R) \to (B, S)$ being those functions $f : A \to B$ such that aRa' implies $f(a)Sf(a')$. Show this category is cartesian closed by describing it as a subcategory of graphs.

7. Consider the category of sets equipped with a distinguished subset, $(A, P \subseteq A)$, with maps $f : (A, P) \to (B, Q)$ being those functions $f : A \to B$ such that $a \in P$ iff $f(a) \in Q$. Show this category is cartesian closed by describing it as a category of pairs of sets.

8. Consider the category of "pointed sets," i.e. sets equipped with a distinguished element, $(A, a \in A)$, with maps $f : (A, a) \to (B, b)$ being those functions $f : A \to B$ such that $f(a) = b$. Is this category cartesian closed?

9. Show that for any objects A, B in a cartesian closed category, there is a bijective correspondence between points of the exponential $1 \to B^A$ and arrows $A \to B$.
10. Show that the category of ωCPOs is cartesian closed, but that the category of *strict* ωCPOs is not (the strict ωCPOs are the ones with initial object $\perp$, and the continuous maps between them are supposed to preserve $\perp$).
11.(a) Show that in any cartesian closed poset with joins $p \vee q$, the following "distributive" law of intuitionistic propositional calculus holds:

$$((p \vee q) \Rightarrow r) \Rightarrow ((p \Rightarrow r) \wedge (q \Rightarrow r))$$

(b) Generalize the forgoing problem to an arbitrary category (not necessarily a poset), by showing that there is always an arrow of the corresponding form.

(c) If you are brave, show that the previous two arrows are isomorphisms.

12. Prove that in a CCC $\mathbf{C}$, exponentiation with a fixed base object C is a contravariant functor $C^{(-)} : \mathbf{C}^{\mathrm{op}} \to \mathbf{C}$, where $C^{(-)}(A) = C^A$.
13. Show that in a cartesian closed category with coproducts, the products necessarily distribute over the coproducts,

$$(A + B) \times C \cong (A \times C) + (B \times C).$$

Do this "directly," i.e. not using the Yoneda Lemma.

14. In the λ-calculus, consider the theory (due to Dana Scott) of a *reflexive domain*: there is one basic type D, two constants s and r of types

$$s : (D \to D) \to D$$
$$r : D \to (D \to D),$$

and two equations,

$$srx = x \quad (x : D)$$
$$rsy = y \quad (y : D \to D).$$

Prove that, up to isomorphism, this theory has only one model M in **Sets**, and that *every* equation holds in M.

15. Complete the proof from the text of Kripke completeness for the positive fragement of IPC as follows:

(a) Show that for any poset I, the exponential poset $\mathbf{2}^I$ is a Heyting algebra. (Hint: the limits and colimits are "pointwise", and the Heyting implication $p \Rightarrow q$ is defined at $i \in I$ by $(p \Rightarrow q)(i) = \top$ iff for all $j \leq i, p(j) \leq q(j)$).

(b) Show that for any poset CCC $\mathbf{A}$, the map $y : \mathbf{A} \to \mathbf{2}^{\mathbf{A}^{\mathrm{op}}}$ defined in the text is indeed (i) monotone, (ii) injective, and (iii) preserves CCC structure.

16. Verify the claim in the text that the products $A \times B$ in categories $\mathbf{Sets}^I$ of I-indexed sets (I a poset) can be computed "pointwise". Show, moreover, that the same is true for all limits and colimits.