

Covariance, Invariance, and Equivalence: a Viewpoint

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Introduction

One of the difficulties encountered when one discusses the principles underlying the general theory of relativity is the lack of agreement on the content of these principles. With a few notable exceptions, most authors agree that a principle of equivalence and a principle of general covariance underlie the theory. Einstein [1], himself, held these principles together with a 'Mach's Principle' to be the basis for general relativity. Beyond this point, however, there is very little agreement.

There are almost as many statements of a principle of equivalence as there are authors [2]. Furthermore, there is no general agreement as to the role of such a principle in the theory. Some authors contend that it is central to the theory; others contend that it is at best an heuristic principle while at least one author would dispense with it entirely. Unfortunately, Einstein nowhere, to our knowledge, stated the principle in precise enough terms to settle the question.

The Principle of General Covariance (hereafter abbreviated PGC) likewise has been called into question. Shortly after Einstein set forth the general theory, Kretchmann [3] wrote a lengthy paper in which he held that the PGC was only a mathematical requirement without physical content. This view has more recently been advanced by Fock [4] who asserts that the term 'general relativity' itself is meaningless. Again, Einstein did not help matters. In replying to Kretchmann, he essentially agreed with his view [1]. And yet, it is clear from all his writings that Einstein held the PGC to be an essential element of general relativity.

Finally, there is Mach's Principle. To the best of our knowledge, Mach never formulated, in precise terms, a principle that can be identified with what it is usually taken to be today. Einstein gave some content to the principle when he wrote [5], 'But in the second place the theory of relativity makes it appear probable that Mach was on the right road in his thought that inertia depends upon a mutual action of matter.' Unfortunately, the term 'depends' is not without ambiguity. Does one mean 'depends solely' or 'depends in part'? Einstein, himself, was later to come to the conclusion

that Mach's Principle was not essential to general relativity. However, the controversy over the content of Mach's Principle and its role in general relativity is still with us; and again, there are as many interpretations of the principle as there are writers on the subject [6].

While one could perhaps have wished that Einstein had been more precise in his formulations of these principles, for him such precision was not necessary, it was the end product that counted. And for the most part it is what counts today—a theory in which the gravitational field and the 'geometry' of space-time are simultaneously characterized by a second-rank symmetric tensor satisfying the famous Einstein equations. Nevertheless, there is still some value in re-examining the underlying principles of general relativity, as indeed there is for any theory. This is especially so in the case of general relativity since, at present, there is no experimental evidence that cannot be equally well explained by a suitable theory of gravity within the framework of special relativity. Differences between the two theories are hopelessly beyond the ability of present day experimental techniques to distinguish. Nevertheless, there are predictions of the general theory, such as the phenomena of gravitational collapse, which, if true, would have a fundamental bearing on our picture of the physical world.

It is our purpose here to re-examine the underlying principles of general relativity, to give what we hope are precise statements of these principles, and to discuss their inter-relation. We do not expect, or even hope, that the formulations given here will be generally accepted; but rather that they will serve as a basis for further discussion. We begin our discussion with a brief review of a previous formulation [7] of the PGC. We then relate this formulation to the idea of the so-called active and passive interpretations of coordinate transformations and finally show how this relation is in accord with one statement of a principle of equivalence.

Covariance

Usually one does not distinguish between covariance and invariance, and indeed the two terms are often taken to be synonymous. Nevertheless, one can make a distinction which, as we shall argue, is critical to the question of the meaning of the PGC. To make this distinction, we first recognize that every physical theory consists of two parts. One of these parts comprises what we have called elsewhere [7], 'the kinematically possible trajectories' (kpt's), that are to be associated with the measurable components of the physical system described by the theory. Thus, if we are dealing with a system of particles, a kpt would consist of a specification of the coordinates of each particle as a function of an appropriate path parameter. At this stage, no restrictions other than kinematical are placed on the values of the coordinates. Of course, we must require that, at least in principle, all of the quantities that characterize a kpt are observable. The second constituent of a physical theory is its dynamical laws or equations of motion that pick out from the totality of all kpt's a subset which we call, 'the dynamically

possible trajectories' (dpt's). The dpt's consist of all those kpt's which, at least in principle, can actually be executed by the physical system being described.

We will now define a *covariance group* of a given physical theory as any group (a) for which the kpt's constitute a faithful realization and (b) which maps a dpt onto a dpt. Our definition is thus a somewhat more formal restatement of the more customary requirement that the form of the dynamical laws of a theory remains unchanged under the action of a covariance group. Given a covariance group, we can divide the dpt's of a theory into *equivalence classes*, two dpt's belonging to the same equivalence class if they can be mapped onto each other by the action of the group. The equivalence classes, therefore, correspond to distinct intrinsic states of the physical system described by the theory.

Given a physical theory, there exists a unique covariance group associated with it. However, the same cannot be said for a physical system. The reason for this lack of uniqueness lies in a lack of uniqueness of the elements that go to characterize the kpt's used to describe the theory. Consider, for example, the motion of a charged particle in a uniform magnetic field. We can use the particle coordinates $x(t)$, $y(t)$, $z(t)$ as the elements of the kpt's and take as the dynamical laws the equations

$$m\ddot{x} = e\dot{y}B \tag{1a}$$

$$m\dot{y} = -e\dot{x}B \tag{1b}$$

$$m\ddot{z} = 0 \tag{1c}$$

where B is a given constant. The covariance group of this theory is then just the group of rotations about the z -axis plus translations along the z -axis and translations in time; that is, 'the group $O'(1) \times T$ '. However, we can also characterize the kpt's by the particle coordinates plus the magnetic field B and take as the dynamical laws

$$m\ddot{r} = e\dot{r} \times B \tag{2a}$$

and

$$\dot{B} = \nabla \cdot B = \nabla \times B = 0, \quad B^2 = B^2 \tag{2b}$$

where B is the same constant that appears in equations (1). Now the covariance group consists of the inhomogeneous rotations in three dimensions plus the group of time translations; that is, 'the group $O'(3) \times T$ '. Nevertheless, it is clear that both theories describe the same physical system with the same intrinsic states since one can establish a one-to-one correspondence between the equivalence classes of the two theories. We can enlarge the covariance group even further, by using curvilinear coordinates, to the group of arbitrary coordinate transformations in three-dimensions plus the time translations. But again, the resultant theory describes the same physical system as that described by the two previous theories since we can again establish a one-to-one correspondence between the equivalence classes of the three theories.

As a second example, consider the conduction of heat in some medium. We can characterize the kpt's by the temperature field $\theta(\mathbf{r}, t)$ and take as the dynamical law the equation

$$\nabla^2\theta - \kappa \frac{\partial\theta}{\partial t} = 0 \quad (3)$$

The covariance group of this theory is the inhomogeneous rotation group in three-dimensions, $O'(3)$. We can also characterize the kpt's by θ and a four-vector field u^μ and take as the dynamical laws

$$(u^\mu u^\nu - \eta^{\mu\nu})\theta_{;\mu\nu} - \kappa u^\mu \theta_{;\mu} = 0 \quad (4a)$$

and

$$\eta_{\mu\nu} u^\mu u^\nu = 1, \quad u^\mu_{;\nu} = 0 \quad (4b)$$

where $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ and a comma denotes ordinary differentiation, i.e., $\theta_{;\mu} \equiv \partial\theta/\partial x^\mu$. The covariance group of this theory is the inhomogeneous Lorentz or Poincaré group P . We can enlarge the covariance group even more by introducing a symmetric 'metric tensor' $g_{\mu\nu}$ in addition to θ and u^μ as components of the kpt's and take as the dynamical laws of the theory

$$(u^\mu u^\nu - g^{\mu\nu})\theta_{;\mu\nu} - \kappa u^\mu \theta_{;\mu} \quad (5a)$$

$$g_{\mu\nu} u^\mu u^\nu = 1, \quad u^\mu_{;\nu} = 0 \quad (5b)$$

and

$$R_{\mu\nu\rho\sigma} = 0 \quad (5c)$$

where $R_{\mu\nu\rho\sigma}$ is the Riemann-Christoffel tensor formed from $g_{\mu\nu}$ and a semicolon denotes covariant differentiation. Now the covariance group of the theory is the group of arbitrary local coordinate transformations in a four-dimension space, the Einstein group E .

From the above examples we see that while a physical theory has associated with it a unique covariance group, a physical system does not. It is just this lack of uniqueness which led Kretschmann and others to conclude that the requirement of general covariance—that is, covariance under the Einstein group E —was devoid of physical content. The central point of Kretschmann's objection to general covariance was that any physical system could be described by a general covariant theory and, therefore, the requirement of general covariance was only a mathematical requirement.

If we examine the above examples, we see that the lack of uniqueness of the covariance group associated with a given physical system is due to a lack of uniqueness in the characterization of the kpts' to be associated with the system. Whether we choose to consider the magnetic field as given *ab initio* as in the first theory of the motion of a charged particle in a uniform magnetic field or include it as an observable part of the system to be included in the description of the kpt's is purely a matter of choice. Likewise, whether we consider the velocity of the conducting medium in the second example to be given or as a measurable component of the system is again a matter of choice. Furthermore, it is just the inclusion of these elements in the kpt's

that leads to the enlargement of the covariance group over that when they are not included.

At first sight, it might appear that one could enlarge indefinitely the covariance group to be associated with a given physical system by introducing enough additional elements into the description of the kpt's associated with the system. However, it is clear that this is not the case since every dpt must, at least in principle, be observable in its entirety. Thus, one must be able to observe *all* of the values of *each* element that characterizes the kpt's predicted by the theory, that is, all values corresponding to the dpt's of the theory. In the example of the charged particle in the uniform magnetic field, if we choose to include \mathbf{B} as a component of the kpt's, then each value of \mathbf{B} allowed by the equations (2b) must be observable. By suitably orienting the reference frame, it is clear that all such values can be observed. Likewise, in the heat conduction example, all values of the four-velocity u^μ allowed by equations (4b) can be observed by a suitable choice of reference frame.

On the basis of the above discussion we argue that the requirement of covariance with respect to a given group is a strong physical requirement if we, at the same time, require that every dpt of the theory that admits the group as a covariance group be observable at least in principle. It then becomes a physical question whether or not a given group is a possible covariance group for a given physical system or class of physical systems. Thus, the PCG is, in principle, a testable hypothesis. Among other things, it would require that every metric satisfying equations (5c) be observable. To the best of our knowledge, no one has suggested that such is not the case.

Although the requirement of covariance, under a group, clearly limits the possible choice of theories that can describe a given physical system, it is not the strongest group theoretical requirement that one can impose. Thus, one could replace equations (5c) by the Einstein equations

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = \kappa T^{\mu\nu} \tag{6}$$

where $T^{\mu\nu}$ is the stress-energy tensor of the conducting medium without changing the covariance group. Nevertheless, the two theories are quite different in content since, among other things, it is not possible to place the equivalence classes of the two theories in one-to-one correspondence. Additional restrictions arise when we require *invariance* under a group. We now turn our attention to the definition and properties of this concept.

Invariance

In the examples of the previous section, the covariance group was enlarged by introducing additional elements into the description of the kpt's. Nevertheless, these quantities, the magnetic field \mathbf{B} in the first example and the four-velocity u^μ and metric $g_{\mu\nu}$ in the second example, differ from the original quantities \mathbf{r} and θ that characterized the kpt's in an

essential way. While the values of these latter quantities change from equivalence class to equivalence class, the values of the former quantities do not. For this reason we have called quantities such as r and θ *dynamical objects* and quantities such as B , u^μ and $g_{\mu\nu}$, as they appear in theories with dynamical laws such as given by equations (2b), (4b) or (5b), (5c), *absolute objects*. Thus, an absolute object has only one intrinsic set of values for all intrinsic states of the system while the intrinsic values of the dynamical values change from state to state and indeed characterize these different intrinsic states. (By intrinsic values we mean here values modulo the covariance group.) More specifically, if $\{y_A\}$ characterizes a kpt and if it is possible to decompose the components y_A into two sets g_x and ϕ_a such that

- (i) the ϕ_a constitute the basis of a faithful realization of a covariance group G and
- (ii) the ϕ_a have the same intrinsic values in every equivalence class of dpt's under G

then the ϕ_a are the components of the absolute object (or objects) of the theory while the remaining g_x constitute the components of the dynamical object (or objects) of the theory. We see that the essential difference between absolute and dynamical objects is that, whereas the values of the latter quantities depend on the values of the former quantities, the converse is not true. Theories with absolute objects, therefore, violate a generalized kind of law of action and reaction. We shall return to this point presently.

We now define the *invariance group* of a physical theory as that subgroup of the covariance group of the theory which leaves invariant the absolute objects of the theory. In particular, if there are no absolute objects, the invariance group and the covariance group are the same group. In the example of the charged particle moving in an external magnetic field, there are no absolute objects when the kpt's are characterized by the particle coordinates alone; and so the invariance group and the covariance group are both $O'(1) \times T$. When the kpt's are characterized by both the particle coordinates and the magnetic field B satisfying equations (2b), B is an absolute object since every B that satisfies these equations can be transformed to have the values $B_x = B_y = 0$, $B_z = B$ by an element of the covariance group $O'(3) \times T$. The subgroup of $O'(3) \times T$ that leaves B invariant is just $O'(1) \times T$. Thus in both theories, the invariance group is the same, namely $O'(1) \times T$. In the example of heat conduction, the situation is the same. When the kpt's are characterized by θ , there are no absolute objects; and so again, the invariance group coincides with the covariance group $O'(3)$. When the kpt's are characterized by θ and u^μ , the latter is an absolute object and can always be transformed by an element of the covariance group P to the values $(1, 0, 0, 0)$. The subgroup of P that leaves u^μ invariant is $O'(3)$. Finally, if the kpt's are characterized by θ , u^μ and $g_{\mu\nu}$, both u^μ and $g_{\mu\nu}$ are absolute objects and can be transformed by an element of the covariance group E to the values $(1, 0, 0, 0)$ and $\text{diag}(1, -1, -1, -1)$ respectively. The subgroup of E that leaves invariant these objects is again $O'(3)$.

From the above examples we see that each physical system has associated with it a unique invariance group. Consequently, the requirement that a theory admits a given group as an invariance group is much more restrictive than the requirement that it admits the group as a covariance group, although we emphasize again that the latter requirement is not empty. Thus the requirement of Poincaré invariance would rule out a theory such as the one discussed above while Poincaré covariance would not. To construct a Poincaré invariant theory of heat conduction, one would have to replace equations (4b) by some other set that would not restrict u^{μ} to being an absolute object. Phenomenological theories of this type have been proposed by Eckart [8] and by Landau and Lifshitz [9].

In addition to the restrictions placed on possible physical theories by an invariance requirement, the invariance group plays an important role in determining the conservation laws associated with a given theory via Noether's theorem [10]. The general result is that, if the dynamical laws governing the dynamical objects are derivable from a variational principle, there is a conserved quantity for each parameter needed to specify an element of the invariance group and that the totality of conserved quantities serves as the basis of a faithful realization of the invariance group. In fact, the totality of conserved quantities constitute the generators of the invariance group. Thus, if we have a Poincaré invariant theory, there will be, in general, ten conserved quantities corresponding to conservation of energy, momentum, angular momentum, and center of energy motion. If, on the other hand, the invariance group is the Einstein group, which requires four space-time functions to specify an element of the group, there will be a four-fold infinity of conserved quantities no subset of which, by themselves, constitutes a realization of the Einstein group [11].

Before we conclude this discussion of invariance, we would like to point out the relation between the so-called 'active' and 'passive' transformations and invariance and covariance. A passive transformation is one in which the state of the physical system is left unchanged while the 'reference frame' is transformed. For an active transformation, the roles of reference frame and physical system are reversed. In our example of the charged particle in the external magnetic field, which we have considered to be unaffected by the motion of the particle, we can realize all of the transformations of the group $O'(3) \times T$ by suitably orienting our orthogonal coordinate system with respect to the external field. All of the particle trajectories and fields allowed by the equations of motion (2a, b) could, in principle, be observed in this way. On the other hand, since the external field is supposed not to be at our disposal to modify, the only active transformations that one can carry out are rotations about and translations along the field; that is, 'the group $O'(1) \times T$ '. By carry out, we mean here, changes of initial conditions in the particle coordinates and velocity resulting in two different trajectories related by an element of the group $O'(1) \times T$. We see from this example (and similar statements can be made for the heat conduction example) that the passive transformations that one can carry out on a physical system corres-

pond to the covariance group of a possible theory that describes the system while the active transformations correspond to its invariance group. We also note that it is impossible, on the basis of observations made on the system alone, to distinguish between an active transformation and the corresponding passive transformation. Whether we rotate our reference frame by a given amount about the magnetic field of our example or rotate the trajectory of the particle by the opposite amount, we obtain the same result as far as the dpt describing the particle trajectory is concerned. The only way we could decide between the two alternatives is by referring to some standard exterior to the system.

General Invariance and the Principle of Equivalence

So far our discussion of covariance and invariance has been quite general. Let us now consider how these concepts apply to the general theory of relativity. We have already asserted that a principle of general covariance is operative in nature, namely that every physical system can be described by a physical theory that admits the Einstein group as a covariance group.

Whether this assertion is correct or not is, of course, open to question; and, indeed, Fock has argued strongly against its validity. According to Fock, a 'preferred' class of coordinate systems, the so-called harmonic systems, exists in which the laws of nature assume a particular simple form. However, what he has not demonstrated is that other coordinate systems are not equally valid for observing the dpt's of a physical system. Only if one is willing to deny this possibility can one object to the principle of general covariance.

Granting the principle of general covariance, one still does not arrive to general relativity since, among other things, the requirement of general covariance would not rule out theories in which the metric is determined by equations of the form (5c). What is needed for this purpose is a principle of *general invariance* whereby we require that the Einstein group be an invariance group of all physical systems. General relativity can then be said to comprise all physical systems that are Einstein invariant, just as special relativity consists of all systems that are Poincaré invariant and classical or Newtonian physics consists of all systems that have the inhomogeneous Galilean group as their invariance group.

If the principle of general invariance holds, then we can rule out all theories in which $g_{\mu\nu}$ is an absolute object. While the principle does not determine what equations $g_{\mu\nu}$ must satisfy, the additional requirements that these equations be of second differential order and are derivable from a variational principle leads almost uniquely to the Einstein equations (6). The only non-uniqueness remaining is the possibility of adding a so-called cosmological term $\lambda g^{\mu\nu}$ to the left side of equations (6). This strong uniqueness of the Einstein equations is to be contrasted with the non-uniqueness of the Maxwell equations. To obtain these latter equations, it is necessary to

add, in addition to the requirements of Poincaré invariance, a variational principle and second differential order, the further requirement of linearity.

Let us now consider what justification one has for accepting a principle of general invariance. Of course, as in all physics, the final justification for any such principle must rest on the observational verification of its consequences. Unfortunately, experimental verifications of general relativity are hard to come by. Nevertheless, there are, in our opinion, strong reasons for accepting a principle of general invariance.

The first of these rests on what might be called a generalized principle of action and reaction. Both the Newtonian world picture and that of special relativity contain absolute objects, the planes of absolute simultaneity and an affine structure in the Newtonian case and the light-cone structure in the case of special relativity. The existence of these absolute objects makes itself manifest when we formulate generally covariant Newtonian or special relativistic theories [12]. But theories with absolute elements suffer from a lack of reciprocity; the absolute objects influence the behavior of the dynamical objects but not vice versa. At least, one should suspect theories with absolute objects since most of our experience with nature seems to point to the absence of absolute objects. If one part of a physical system can influence another, then the reverse influence also seems to be present. Whether this reciprocity is a universal feature of all physical system is, of course, open to question. If, indeed, it exists, we would then have operative what we have called a generalized principle of action and reaction.

It is even possible to argue that such a principle is really the essence of Mach's Principle. The inertial properties of matter in both Newtonian physics and special relativity can be said to be a consequence of the absolute objects in these theories. If, however, we accept the Machian view that these inertial properties 'depend on the mutual action of matter', then we are forced to consider these objects no longer as absolute but rather dynamical. In any event, we see that a denial of the existence of absolute objects together with the requirement of general covariance of all physical laws leads directly to the principle of general invariance as we have formulated it.

Stronger support for the principle of general invariance comes from the Principle of Equivalence. Usually the principle is taken to assert the indistinguishability of gravitational and inertial effects, at least locally. The strongest experimental support for this assertion is the well-known experiments of Eötvös [13] and Dicke [14] that have established the so-called equivalence of inertial and passive gravitational mass to better than one part in 10 [11] in the case of the Dicke experiments. The bending of light in the gravitational field of the sun also appears to support this version of the principle.

Nevertheless, it is possible to construct theories that are not in accord with this statement of the principle of equivalence and yet, in every other regard, can be called general relativistic theories. Such a theory can be constructed by adding to the action for the combined Einstein-Maxwell equations, a term proportional to $\int \sqrt{-g} R_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} d^4x$. If such a term

is present, then regardless of how small a space-time region one considers it would not be possible to reduce the equations for $F_{\mu\nu}$ to their special relativistic form by a suitable choice of reference frame. Consequently, one could, with the help of such a field, distinguish gravitational effects from inertial effects but not vice versa. It would, however, still be possible to construct gravitational fields that locally were indistinguishable in their action from inertial effects. We need only require that locally $R_{\mu\nu\rho\sigma}$ vanish for these fields. However, it would not be possible to reproduce the gravitational effects by inertial effects when $R_{\mu\nu\rho\sigma}$ did not vanish.

The requirement of complete equivalence of inertial and gravitational effects is seen to be a very restrictive requirement on the interaction of matter with the gravitational field. For this reason, we have referred to it elsewhere as the principle of minimal coupling. Dicke [15] has referred to it as the 'strong' principle of equivalence. But, unlike Dicke and other authors, we do not hold that such a principle is essential to general relativity; it is only a useful principle of selection that may or may not be valid. What is essential to general relativity is the requirement that all inertial effect can be duplicated by gravitational effects. This 'weak' principle of equivalence (not to be confused with Dicke's weak principle of equivalence) was formalized by Trautman. According to him, to the extent that we can neglect their action back on the sources of the gravitational field present in a given space-time region, measurements made on any physical system will serve to determine the same affinity in this region. By affinity, we mean here that quantity which must be used to convert ordinary derivatives into covariant derivatives in the covariant form of the dynamical laws of the system. This weak form of the principle allows the possibility of systems that are not coupled minimally to the gravitational field while, at the same time, it retains the essential content of equivalence, namely that inertial effects are indistinguishable from gravitational effects.

If we accept this weak form of the principle of equivalence, we are immediately led to the conclusion that it is impossible to distinguish between arbitrary passive and active coordinate transformations since passive coordinate transformations result in inertial effects while active coordinate transformations, among other things, produce changes in the gravitational field. It is, we contend, just this impossibility of distinguishing between active and passive coordinate transformations that is implied by the term 'general relativity', just as the impossibility of distinguishing between active and passive Poincaré transformations is implied by the term special relativity and the impossibility of distinguishing between active and passive inhomogeneous Galilean transformations by Galilean relativity. But since the passive transformations comprise the covariance group of a theory and the active transformations comprise the invariance group, the indistinguishability of active and passive coordinate transformations implies that the Einstein group is an invariance group for all physical systems. Thus the validity of the weak principle of equivalence leads directly to the principle of general invariance.

Conclusion and Summary

We have given what, we hope, are precise statements of the main principles that underly general relativity. It is not our intention to argue that these statements correspond to the original statements of these principles. We have used the original terms, principle of equivalence, etc., to designate the formulations given here primarily for convenience and because we feel that our formulations are, at least, in the spirit of the general accepted meaning of these appellations. Our formulation of these principles rests primarily on the observation that there are two types of objects that appear in the descriptions of physical systems, which we have termed absolute and dynamical. A covariance group consists of those transformations which transform one observable set of values of these objects into another observable set and correspond to the passive transformations that one can carry out on the system. An invariance group is the subgroup of a covariance group that leaves invariant the absolute objects. If there are no absolute objects, the two groups coincide.

The principle of general covariance asserts that the Einstein group, that is, 'the group of arbitrary local coordinate transformations', is a covariance group of all physical systems that can be described by space-time theories. The requirement that the Einstein group is also an invariance group of all physical systems constitutes the principle of general invariance. Support for this latter principle comes first from a Machian type of argument that there should be no absolute objects in nature. A weak form of the principle of equivalence that asserts the equivalence of inertial effects to gravitational effects (but not necessarily conversely) allows one to conclude the impossibility of distinguishing between arbitrary active and passive coordinate transformations. Since the latter transformations constitute the Einstein group, we conclude that the former also constitute this group. Since the active transformations constitute an invariance group, we are again led to the conclusion that the Einstein group is the invariance group of all space-time systems.

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