

**On the Physical Meaning of Relativity Principles,
A. Einstein's New and his Original Theory of Relativity**

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[translated from the German by Robert Rynasiewicz,
with the assistance of Frank Döring]

Contents: Introduction. Subject and results of the project. — I. On the physical meaning of relativity principles. — II. On the measurability in principle of the components $g_{\mu\nu}$ of Einstein's gravitational potential. III. Limiting the covariance of the Einstein equations. 1. Employment of the “directional axes” of the curvature tensor as coordinate directions. 2. On the introduction of absolute invariants as space and time coordinates. 3. Closer determination of reference systems by the conditions imposed on $g_{\mu\nu}$. — IV. Geometrical determination of the relativity principles satisfied by Einstein's new theory of relativity and comparison with the original theory of relativity. — Conclusion: On the reason for the unsatisfiability of the general principle of relativity.

Introduction.

The manner in which various authors have expressed the principle of the Lorentz-Einstein theory of relativity,¹ and especially Einstein's recent formulation of his general principle of relativity,² directly suggests or — as is the case with Einstein — demands that a system of physical laws satisfies a relativity principle if the equations by which it is characterized are covariant under the transformation group of space and time coordinates associated with the principle.³ If one accepts this interpretation and keeps in mind that all physical observations consist ultimately in the determination of purely topological relations (“coincidences”⁴) between observable spatio-temporal objects, and therefore no coordinate system is directly granted privilege over any other

¹Cf., e.g., H. Minkowski: “Raum und Zeit”. B. G. Teubner 1909. p. 4. — M. v. Lauer[sic]: “Das Relativitätsprinzip”. Vieweg 1911. p. 33, § 6. — M. Abraham: “Theorie der Elektrizität”. B. G. Teubner 1908. p. 379 and 380.

²A. Einstein, *Ann. d. Phys.* **49**. p. 776. 1916.

³I have used the words “relativity principle” [“Relativitätspostulat”] and “theory of relativity” [“Relativitätstheorie”] in the sense of this interpretation also in my article: “Über die prinzipiell Bestimmbarkeit usw.”, *Ann. d. Phys.* **48**. p. 907–982; cf. loc. cit. p. 910, note 5. In fact, though, this is obviously not the meaning for the topic treated there.

⁴A. Einstein, loc. cit.

by them,⁵ then one will be forced to the conclusion that any physical theory can be brought into agreement with any relativity principle whatsoever — including the general — by means of a purely mathematical, although perhaps extremely complicated reformulation of the characterizing equations, without altering its observationally testable content — whatever it may be.⁶

Nonetheless, it must surely be possible to ascribe to relativity principles yet another meaning other than this purely formal mathematical one. For only from the existence of such can it be explained, for example, why, as is evident and generally recognized, it is impossible to introduce into Einstein's original theory the concept of a rigid body, although this can be defined by purely topological criteria as easily as almost no other.⁷

The first part of the present work attempts to develop this physical meaning of relativity principles in an example by means of four-dimensional geometric considerations and [then], in generally valid terms, to explicate the significance of an arbitrary relativity principle. The application of the resulting concept of relativity — differing essentially from Einstein's — to Einstein's new theory of gravity then leads in the following sections to the result (§ 25), that the theory is to be regarded in its physical content as a completely absolute theory, which in substance satisfies no relativity principle whatsoever. In contrast, Einstein's original theory of relativity proves to be the widest that is at all conceivable given certain general assumptions (§ 26).

In conclusion it will be shown that the general principle of relativity could be fulfilled physically, in the sense of the interpretation taken up here, only by laws of nature whose general character — unconditionally affirmative [unbedingt bejahend] — would differ fundamentally from that — conditional, i.e., negative [bedingten, d. h. verneinenden] — of the laws so far proposed.

As already noted in the foregoing, the contrast, which arises between the present work and the viewpoint articulated by Einstein in his investigations in gravitational theory, rests in my opinion entirely on a certainly important difference in the interpretation and conceptual analysis of relativity principles. The contrast concerns only the classification of Einstein's "general" theory and of his original theory of relativity in the hierarchy of generally conceivable theories of relativity. However, the question as to the actual validity of the laws of nature newly proposed by Einstein remains entirely unaddressed.

I. On the Physical Meaning of Relativity Principles

§1. Einstein based his postulation of the general covariance of the equations of physics under arbitrary continuous coordinate transformations⁸ essentially

⁵For more on this: E. Kretschmann, loc. cit. p. 914–924.

⁶Cf. G. Ricci et T. Levi-Civita: Méthodes de calcul différentiel absolu et leurs applications". Math. Ann. **54**. p. 125. 1901.

⁷Cf. E. Kretschmann, l.c. p. 967 and 968 § 55.

⁸A. Einstein: "Die Grundlagen der allgemeinen Relativitätstheorie", Ann. d. Phys. **49**. p. 769–822. 1916; cf. p. 776 and 777.

on the fact that all physical experience consists ultimately in the observation of purely topological relations or “coincidences” between spatio-temporal objects of observation. Thus there is no ground in experience for granting privilege to any reference system for space and time over all others as the only correct one. Consequently, for example, the reference system Σ ($x_1 = x$, $x_2 = y$, $x_3 = z$, $x_4 = ict$), in which, to the extent it holds at all, the well-known light propagation equation

$$(1) \quad (x_1 - x_1^0)^2 + \cdots + (x_4 - x_4^0)^2 = 0$$

of Einstein’s original theory of relativity is satisfied, in fact has no privilege over any other system of reference.

As a result does the original theory of relativity as such now lose all physical content? It seems to me this is hardly the case.

§2. For any given system of reference there is, according to the original theory of relativity, a well-defined class of equally admissible [gleichberechtigter] ones in which the laws of physics assume the same mathematical form. The rectilinear, orthogonal reference systems, in which the light propagation equation has the form (1), arise from one another by the transformations of the group which one obtains by combining the Lorentz group with the translation group, $x_1' = x_1 + a_1, \dots, x_4' = x_4 + a_4$, and the group of uniform dilations $x_1' = \lambda x_1, \dots, x_4' = \lambda x_4$. If one switches to another reference system, e.g., polar coordinates, then not only the form of equation (1) changes but also the form of the “admissible” [“berechtigten”] transformations which leave it invariant.

However, the group which comprises the admissible transformations in question always remains “similar” [“ähnlich” — i.e., isomorphic] to the original. It shares with it all those properties [that are] entirely independent of the choice of reference systems (e.g., the number of parameters, etc.). These are the only ones that are essentially group-theoretic in character so that, following Lie, one is accustomed to collecting together all similar groups under the concept of an (invariance) group. Using this concept of an invariance group, one obtains, at least provisionally, the following general formulation for an arbitrary (special) relativity principle: The laws of physics are — regardless of which coordinates they are written in — covariant with respect to the (invariance) transformation group G .

Here, by G , is to be understood a group uniquely associated with the relativity principle in question. We have thus found a formulation completely independent of the choice of reference system, even for relativity principles which do not demand general covariance.

§3. Nonetheless, the physical content of the relativity principle belonging to an invariance group G is still not fully apparent in the form given. One can see this easily in the example of the light equation (1). For it is easy to put this equation into a form [that is] invariant with respect to entirely arbitrary coordinate transformations without in any way altering its physical content. One needs only to introduce the manner of expression of the general theory of

relativity and to write in place of equation (1):⁹

$$(2) \quad \begin{cases} \delta \int ds = 0, \\ ds^2 \equiv \sum_{\mu\nu} g_{\mu\nu} \cdot dx_\mu \cdot dx_\nu = 0, \\ (\lambda\nu, \mu\tau) \equiv 0 \end{cases} \quad (\lambda, \nu, \mu, \tau = 1 \dots 4).$$

Here, ds denotes the invariant line element and $(\lambda\nu, \mu\tau)$ the components of the Riemann curvature tensor of the space-time manifold,¹⁰ by the identical vanishing of which the laws of Einstein's new relativity theory reduces to the original.

According to the general characterization of a relativity principle given above it would have to follow that these very same light propagation laws, depending on the form of their representation, satisfy on the one hand the general principle of relativity — when represented by (2) — and on the other only a special principle of relativity in the other form (1). The same would hold for all physical laws. For, according to the investigations of Ricci and Levi-Civita,¹¹ it can hardly be doubted that one can put any system of physical equations into a generally covariant form without altering their observationally testable content. This is clear from the outset if one recalls that, strictly speaking, only purely topological facts about natural occurrences, or, following Einstein, coincidences are observable.

Thus, should the claim, that a system of equations satisfies a special, and no wider a relativity principle, acquire an actual physical, and not merely a formal mathematical significance, then the general concept of a relativity principle must be defined in such a way that, accordingly, one and only one relativity principle — equivalently, the corresponding invariance group — can be determined for any given system of physical laws as the widest that the laws fulfill solely on the basis of their topological content independently of the form of presentation adopted.

§4. That this is possible, and how it is, one again discerns most easily from the light propagation equation of the original relativity theory. It will be assumed in this case, as we will always do in what follows for whatever system of laws is under consideration at the time, that it is satisfied in actuality insofar as its observational content is concerned. If we begin again with form (1) of the law, we see first that among the actual reference systems, which are to be thought of as given by measuring rods and clocks with completely precise length and time measurements, those [systems] in which equation (1) is everywhere and always satisfied are distinguished from all those in which this is not the case by the observational results expected in them in principle, i.e., abstracting from all technical difficulties. For in each of the latter mentioned [systems],

⁹Cf., e.g., A. Einstein u. M. Großmann: "Entwurf einer verallgemeinerten Relativitätstheorie ...". B. G. Teubner 1913. p. 6 § 2 and p. 8 § 3.

¹⁰S. Christoffel, Crelles Journal **70**. p. 54.

¹¹G. Ricci et T. Levi-Civita: Méthodes de calcul différentiel absolu et leurs applications". Math. Ann. **54**. p. 125. 1901.

at least one among all those theoretically possible light pulses passing through the aether sometime and somewhere traces out a world line on which at least one pair of points has coordinate differences which do not stand in the relation required by (1). And provided that the measuring rods and clocks which realize the reference system in question by permitting the coordinates of the world points adjacent to them to be read off are present at the correct moment, something that is always possible in principle, then each such deviation can be ascertained by sheer scale readings, and thus by purely topological observations, and [hence] the reference system in which they occur [can be] eliminated as inadmissible according to (1). [In the same way] as [this holds] for equation (1), the corresponding obviously holds for all equations or differential equations relating the coordinates of physically perceptible entities. Of course, even with completely precise measuring instruments, the derivatives of the coordinates can be measured only approximately; but since the precision of measurement is in principle unlimited, any failure of the reference system [to be admissible according to the differential equation in question] is detectable in this case as well.¹²

§5. Thus, equation (1) and the equations arising from it by means of coordinate transformations pick out [auszeichnen], each for itself, the class of reference systems in which it is satisfied, and thus, all together, the invariance group whose transformations connect the reference systems of a class with one another. If, however, one writes the same law of light propagation in a form different from all these, then the group distinguished by it and those equations into which it transforms will be different. E.g. if instead of (1) we write:

$$(3) \quad \begin{cases} (x_1 - x_1^0)^2 + \dots + (x_3 - x_3^0)^2 + c^2(x_4 - x_4^0)^2 = 0, \\ c = \text{Const.} \end{cases}$$

then, because of the indeterminacy of c , this characterization admits, in addition to the transformations of the group of (1), also the transformation group

$$x_4' = \mu \cdot x_4, \quad x_1' = x_1, \quad x_2' = x_2, \quad x_3' = x_3.$$

The group belonging to (2) includes all continuous transformations whatsoever, and, if one adds to (2) further the auxiliary conditions $g_{\mu\nu} = 0$ for $\mu \neq \nu$, then one has a system of equations physically equivalent to the preceding, but which is invariant with respect to all transformations

$$x_1' = f_1(x_1 \dots x_4) \dots x_4' = f_4(x_1 \dots x_4),$$

which satisfy the restriction:

$$g_{\mu\nu} = 0 = \sum_{\alpha\beta} \frac{\partial x_\alpha'}{\partial x_\mu} \cdot \frac{\partial x_\beta'}{\partial x_\nu} g_{\alpha\beta}' = \sum_{\alpha} \frac{\partial x_\alpha'}{\partial x_\mu} \cdot \frac{\partial x_\alpha'}{\partial x_\nu} g_{\alpha\alpha}'$$

¹²Except in special cases, the demonstration of the converse, that a reference system, i.e., the reports of the corresponding measuring instruments, is consistent with a given equation, can be carried out only by an infinite sequence of measurements. (Cf. E. Kretschmann, *Ann. d. Phys.* **48**, p. 943–959.)

$$(\mu, \nu, \alpha, \beta = 1 \dots 4); \mu \neq \nu,$$

and [which] form again a different group. These examples can be multiplied at will.

What is wanted, however, is a transformation group which is determined solely by the physical content of the laws, independently of the manner of expression adopted.

§6. One is led directly to such [a group] if, for each system of equations, one imagines all the world lines of light impulses consistent with [those equations] drawn out in the multifarious coordinates and compares with one another the four-dimensional geometric pictures thus generated.

All world lines of light rays that are possible according to (1) are obtained if, starting from each world point $x_1^0 \dots x_4^0$, one draws all sides of the forward cone [Nachkegels] [in conformity] with equation (1).¹³ Likewise, each equation obtained by a transformation of (1) describes an infinite class of world lines of light rays which differ in general from the first in comparisons of measure [Maßverhältnissen], but, apart from this, are completely equivalent in all their topological properties. According to (3), however, from each world point emanates instead not one, but an infinite number of cones, namely one for each value of the undetermined constant c remaining. In (2) and in the form of presentation that results from it by conjoining the condition $g_{\mu\nu} = 0$ for $\mu \neq \nu$, there even occur undetermined functions — the ten $g_{\mu\mu}$, [*sic*: this should be $g_{\mu\nu}$], respectively, the four $g_{\mu\mu}$ — instead of the undetermined constant c , and the multiplicity of cones with the same vertex will be correspondingly greater. Here, of course, the sense of the equations is not that in actuality multiple [auch nur zwei — lit.: even as many as two] light waves forming spatially disjoint and closed surfaces can originate from the same point in the aether at the same time. In each case, rather, each single class of world lines which one obtains by a complete (numerical) specification of the coordinate functions $g_{\mu\nu}$ — which though undetermined are unequivocal — or, respectively of the constant c , is a complete picture of the simultaneous and successive light trajectories possible in actuality. Each of these individual pictures, of course, is in perfect topological agreement with the class of world lines determined by (1), and, consequently, the totality of pictures given by the other three systems of equations are compilations of the infinitely many pictures given by (1) and the equations transformed from (1) [resulting from (1) via coordinate transformations — trans. note]. As mentioned [earlier], any one of these suffices for the representation of what in actuality is possible according to the law in question. Any further one is entirely superfluous from a physical point of view.¹⁴

But on the other hand, the particular class of world lines of light rays which

¹³The absolute value of the imaginary coordinate x_4 is to be imagined plotted.

¹⁴On the other hand, the inclusion of an entire class of physically equivalent pictures in the way [the laws are] represented offers mathematical advantages in certain circumstances. (Cf. § 24 in what follows).

is represented in (1), or an equation transformed from (1), cannot in general be [further] reduced without thereby introducing a new law of light propagation going beyond (1). For among the light phenomena at all possible according to (1), there is obviously none which, solely in virtue of equation (1), excludes any other as incompatible with it. It follows from this that each system of equations which gives expression, in whatever form, to exactly the laws contained in (1) must describe in full at least one of the class of world lines of light rays belonging to equation (1) and its transformed forms as an image of possible and mutually consistent light motions, and accordingly [each such system of equations] must be satisfied in every reference system in which this picture, i.e., the pertinent equation transformed from (1), is valid. Consequently, the invariance group of transformations which connects these reference systems is the narrowest which is physically distinguishable from all others by means of any manner of representation of the of the laws contained in (1). It is, as required, determined solely by the physical content of the laws independently of the manner of their expression, and in fact it is, according to what was shown, obviously the unique group for which this holds. Geometrically, the group is characterized as the group of transformations which map [back] onto itself the class of all world lines of light rays which are compossible in a single coordinate manifold according to the laws. Thus each system of equations equivalent in content to (1), (2), and (3) satisfies the relativity principle belonging to this group and no additional [relativity principle].

§7. The preceding determination of the relativity principle satisfied by the [particular] laws of light propagation assumed [there] can be applied straightforwardly to arbitrary systems of physical laws. In doing so, one needs only, in order to do justice to all possibilities, to consider further the fact that the size of the physically distinguished transformation group depends also on “contingent” physical circumstances [that are] not lawfully determined.¹⁵ Nevertheless, it is obvious that a relativity principle can hold as (generally) fulfilled only if it is satisfied independently of such circumstances, which in principle can be altered at will.

Suppose one calls a reference system “physically admissible” [berechtigten] for a given form of representation of a system of laws¹⁶ if it is compatible with it in regard to all observations which are not excluded by the system of laws given

¹⁵Such a case arises in Einstein’s gravitational theory to the extent that, according to it, in the case of certain anomalous [singulären] curvatures of the space-time manifold, which, according to Einstein, depend on the contingent distribution and motion of matter, the physically distinguished group includes exceptionally more transformations than is usual; cf. § 25.

¹⁶The transformations between the “physically admissible” reference systems need not, in each case, also render the corresponding system of equations mathematically invariant. Thus, the well know form $\text{Div } \mathfrak{T} = 0$ of the conservation law of energy and momentum remains invariant only under linear transformations, without this by itself permitting it to be shown by means of physical observations that any reference system is physically inadmissible. (Cf. E. Kretschmann, Ann. d. Phys. **48**, p. 932. 1915.)

and are thus regarded as possible in principle.¹⁷ Then roughly the following general formulation results:

A system of physical laws satisfies the relativity principle of an invariant transformation group G just in case, if for any arbitrarily formed representation of all the laws of the system and only these laws, the reference systems physically admissible [berechtigten] — and thereby those which are in principle distinguishable from all the rest by observation — form under all circumstances physically possible according to the laws a class of such a size that the class of transformations connecting them properly contains or is equal to the group G in some form.

According to this, the validity or invalidity of a relativity principle for a system of physical laws is completely independent of its mathematical form of expression and is determined solely by its physical content.

§8. In the following we will henceforth investigate which relativity principle, so understood, Einstein's "general theory of relativity" satisfies. In doing so, it should not go unmentioned that Mr. Einstein understands obviously something quite different than I do by a relativity principle; for according to him, a physics will be "in accord with the general principle of relativity" if the following postulate is satisfied:

"The general laws of nature are to be expressed by equations which hold for all coordinate systems, i.e., which are covariant with regard to arbitrary substitutions (generally covariant)."¹⁸

Thus here the interpretation of general relativity is thus directed immediately and solely at the expression of the laws of nature, the direct influence of which is excluded in my version of a relativity principle. According to the latter, a relativity principle is satisfied only if the relativity of the reference systems following from it is *necessary* and avoidable in no manner of expression of the laws of nature, while according to Einstein, the general principle of relativity is already satisfied if the general equivalence of all systems of reference is only *possible* and employed in the expression [of the laws].

II. On the Measurability in Principle of the Components $g_{\mu\nu}$ of Einstein's Gravitational Potential

§9. In order to find the covariance properties that are associated in an essential way with Einstein's theory, independently of the choice of expression, it will be attempted to put it into the least covariant form possible without altering its physical content. It suffices if this is done with a portion of the Einstein equations, and for this purpose the laws of motion for light rays and point masses in a gravitational field, on which the whole theory is based, will be

¹⁷As a criterion of possibility in principle, here throughout is to be used only the system of laws considered at the time, and, as mentioned, always supposing them correct in their content. An unconditional criterion of possibility could be given only with unconditionally secure knowledge of the laws of nature.

¹⁸A. Einstein, Ann. d. Phys. **49**. p. 776. 1916

chosen. In the same way in which the light propagation equation of the original theory of relativity could be put into generally covariant form by introducing the undetermined coefficients $g_{\mu\nu}$ into the expression for the line element, so conversely the conversion of the Einstein equations into a less covariant form of equivalent physical content is a matter of fixing either all or a portion of those parts of the determination of the coordinate functions $g_{\mu\nu}$ which depend solely on the choice of reference system. In order to know, in turn, to what extent the reference systems which satisfy the relevant conditions on the $g_{\mu\nu}$'s are distinguished from the rest not only mathematically, but also observationally, which alone decides the validity of relativity principles as they are understood here, one must above all investigate which assertions concerning the values of $g_{\mu\nu}$ can be verified by observations in an empirically given reference system, according to Einstein's theory. This shall be done in what follows.

§10. Let $\gamma_{\mu\nu}(x_1 \dots x_4)$ be the values which are assigned to the $g_{\mu\nu}(x_1 \dots x_4)$ in a reference system $\Sigma(x_1 \dots x_4)$ realized by completely precise measuring instruments, and suppose that their agreement with the values of the $g_{\mu\nu}(x_1 \dots x_4)$, which in Σ satisfy the equations of general relativity, is to be verified. The deviation perhaps arising between the $\gamma_{\mu\nu}$'s and the $g_{\mu\nu}$'s can always be determined by observation if upon substituting the $\gamma_{\mu\nu}$'s for the $g_{\mu\nu}$'s in the equations of motion for light or for a point mass subject only to gravitational forces, these [equations] are not satisfied in $\Sigma(x_1 \dots x_4)$. For apart from those [quantities] only quotients of coordinate differentials occur in these equations, which [quotients] can be determined with arbitrary approximation with the measurement instruments given.

The directions $dx_1 : dx_2 : dx_3 : dx_4$ of the world lines of light rays originating from an arbitrary world point satisfy the following equations according to the general theory of relativity¹⁹:

$$ds^2 \equiv \sum_{\mu,\nu}^{1\dots 4} g_{\mu\nu} dx_\mu dx_\nu = 0$$

or:

$$(4) \quad 0 = \sum_{\mu,\nu}^{1\dots 4} g_{\mu\nu} \frac{dx_\mu}{dx_4} \cdot \frac{dx_\nu}{dx_4}.$$

In order for the difference between the $\gamma_{\mu\nu}$'s and the $g_{\mu\nu}$'s to be undetectable, then along with (4) it must generally hold that:

$$(5) \quad 0 = \sum_{\mu,\nu}^{1\dots 4} \gamma_{\mu\nu} \frac{dx_\mu}{dx_4} \cdot \frac{dx_\nu}{dx_4}.$$

¹⁹A. Einstein, "Die Grundlage der allgemeinen Relativitätstheorie". Ann. d. Phys. **49**, p. 769–822, 1916; cf. p. 777, 778 and 790 ff. § 9.

In order to obtain from this the relations between the $\gamma_{\mu\nu}$'s and the $g_{\mu\nu}$'s, one sets:

$$\text{a) } g_{\alpha\beta}' = \sum_{\mu\nu} u_{\alpha}^{\mu} u_{\beta}^{\nu} g_{\mu\nu},$$

$$\text{b) } \gamma_{\alpha\beta}' = \sum_{\mu\nu} u_{\alpha}^{\mu} u_{\beta}^{\nu} \gamma_{\mu\nu},$$

$$\text{c) } \sum_{\mu} dx_{\mu}' u_{\mu}^{\alpha} = dx_{\alpha}$$

$$(\alpha, \beta, \mu, \nu, = 1 \dots 4)$$

and chooses the 16 quantities u_{α}^{μ} so that the determinant $|u_{\alpha}^{\mu}| \neq 0$ and it holds that:

$$g_{\alpha\beta}' = 0 \text{ and } \gamma_{\alpha\beta}' = 0$$

for $\alpha \neq \beta$.²⁰ Then (4) and (5) become:

$$(4') \quad 0 = \sum_{\alpha} g_{\alpha\alpha}' \left(\frac{dx_{\alpha}'}{dx_4'} \right)^2 \equiv g_{11}' \left(\frac{dx_1'}{dx_4'} \right)^2 + \dots + g_{33}' \left(\frac{dx_3'}{dx_4'} \right)^2 + g_{44}'.$$

$$(5') \quad 0 = \sum_{\alpha} \gamma_{\alpha\alpha}' \left(\frac{dx_{\alpha}'}{dx_4'} \right)^2 \equiv \gamma_{11}' \left(\frac{dx_1'}{dx_4'} \right)^2 + \dots + \gamma_{33}' \left(\frac{dx_3'}{dx_4'} \right)^2 + \gamma_{44}'.$$

We have²¹:

$$g_{11}' \cdot g_{22}' \cdot g_{33}' \cdot g_{44}' = |g_{\alpha\beta}'| = |u_{\alpha}^{\mu}|^2 \cdot |g_{\mu\nu}|$$

and likewise

$$\gamma_{11}' \cdot \gamma_{22}' \cdot \gamma_{33}' \cdot \gamma_{44}' = |\gamma_{\alpha\beta}'| = |u_{\alpha}^{\mu}|^2 \cdot |\gamma_{\mu\nu}|.$$

If $|g_{\mu\nu}|$ vanished in a finite coordinate region, then, in virtue of²²

$$|g_{\mu\nu}'| = \left| \frac{\partial x_{\alpha}'}{\partial x_{\sigma}'} \right|^2 \cdot |g_{\mu\nu}|,$$

²⁰This is possible; since according to a), b), c) the u_{α}^{μ} 's represent the coefficients of the most general single-valued transformation in the infinitely small in which the $g_{\mu\nu}$'s and $\gamma_{\mu\nu}$'s are transformed as components of covariant tensors. Obviously now, one can obtain by an intermediate transformation first $g'_{11} = g'_{22} = g'_{33} = g'_{44} = 1$ and $g'_{\alpha\beta} = 0$ for $\alpha \neq \beta$, and thus bring the axes of the reference system into congruence with the principle axes of the tensor γ by means of a Lorentz transformation, so that also $\gamma'_{\mu\nu} = 0$ for $\mu \neq \nu$ without change of the $g'_{\mu\nu}$ -values.

²¹Cf. A. Einstein, loc. cit. p. 788 and 789. Since for reasons of symmetry I assume that the 4th coordinate x_4 is imaginary, it is the case that $|g_{\mu\nu}| > 0$ and $|\gamma_{\mu\nu}| > 0$ instead of < 0 as in Einstein. [Translator's Note: Since this last clause contains an obvious redundancy, one of the occurrences of 'g' should be primed.]

²²Cf. the footnote before last.

infinitely many coordinate points in general would have to be assigned to each world point of the region. This is to be excluded and is also in principle demonstrable by observation, so that one may suppose at the outset also that $|\gamma_{\mu\nu}| \neq 0$ — apart from singularities.

As a result, none of the quantities $g_{\alpha\alpha'}$ and $\gamma_{\alpha\alpha'}$ is equal to zero, and it must be possible to choose at each coordinate point a generally non-vanishing quantity λ which does not depend on the quotient $dx_{\alpha'}/dx_4'$ so that for arbitrary values of

$$\frac{dx_1'}{dx_4'}, \frac{dx_2'}{dx_4'}, \frac{dx_3'}{dx_4'} :$$

it is the case that

$$0 = \sum_{\alpha} (\gamma_{\alpha\alpha'} - \lambda g_{\alpha\alpha'}) \cdot \left(\frac{dx_{\alpha'}}{dx_4'} \right)^2.$$

By differentiating for $(dx_1'/dx_4')^2$ etc., it follows from this that

$$\gamma'_{\alpha\alpha} = \lambda g'_{\alpha\alpha} \quad (\alpha = 1 \dots 4)$$

and by solving equations (a) and (b) for $g_{\mu\nu}$ and $\gamma_{\mu\nu}$:

$$(6) \quad \gamma_{\mu\nu} = \lambda g_{\mu\nu} \quad (\mu, \nu = 1 \dots 4).$$

The $\gamma_{\mu\nu}$'s must then be proportional to the $g_{\mu\nu}$'s at every coordinate point. The proportionality factor λ is so far, apart from being non-null, an entirely undetermined coordinate function.

§11. The equations of motion of a material point in a gravitational field leads to the further determination of λ . Following Einstein²³, these can be written

$$(7) \quad \sum_{\alpha} g_{\alpha\sigma} \frac{d^2 x_{\alpha}}{ds^2} + \sum_{\mu\nu} \left[\begin{matrix} \mu\nu \\ \rho \end{matrix} \right]_g \frac{dx_{\mu}}{ds} \frac{dx_{\nu}}{ds} = 0,$$

where

$$\left[\begin{matrix} \mu\nu \\ \rho \end{matrix} \right] \equiv \frac{1}{2} \left(\frac{\partial g_{\mu\rho}}{\partial x_{\nu}} + \frac{\partial g_{\nu\rho}}{\partial x_{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x_{\rho}} \right).$$

If one introduces $\gamma_{\mu\nu} = \lambda g_{\mu\nu}$, then:

$$d\sigma \equiv \sqrt{\sum_{\mu\nu} \lambda_{\mu\nu} dx_{\mu} dx_{\nu}} = \sqrt{\lambda} \cdot ds,$$

$$\begin{aligned} \frac{dx_{\alpha}}{ds} &= \frac{dx_{\alpha}}{d\sigma} \cdot \sqrt{\lambda}, & \frac{d^2 x_{\alpha}}{ds^2} &= \lambda \frac{d^2 x_{\alpha}}{d\sigma^2} + \frac{1}{2} \frac{d\lambda}{d\sigma} \cdot \frac{dx_{\alpha}}{d\sigma} \\ \left[\begin{matrix} \mu\nu \\ \rho \end{matrix} \right]_g &= \frac{1}{\lambda} \left[\begin{matrix} \mu\nu \\ \rho \end{matrix} \right]_{\gamma} - \frac{1}{2\lambda^2} \left(\gamma_{\mu\rho} \frac{\partial \lambda}{\partial x_{\nu}} + \gamma_{\nu\rho} \frac{\partial \lambda}{\partial x_{\mu}} - \gamma_{\mu\nu} \frac{\partial \lambda}{\partial x_{\rho}} \right) \end{aligned}$$

²³A. Einstein, loc. cit. p. 791, eqns. (20d) and (21).

$$\text{with } \left[\begin{array}{c} \mu\nu \\ \rho \end{array} \right]_{\gamma} \equiv \frac{1}{2} \left(\frac{\partial \gamma_{\mu\rho}}{\partial x_{\nu}} + \frac{\partial \gamma_{\nu\rho}}{\partial x_{\mu}} - \frac{\partial \gamma_{\mu\nu}}{\partial x_{\rho}} \right).$$

Equation (7) thereby becomes:

$$(7a) \quad \left\{ \begin{array}{l} \sum_{\alpha} \gamma_{\alpha\rho} \frac{d^2 x_{\alpha}}{d\sigma^2} + \sum_{\mu\nu} \left[\begin{array}{c} \mu\nu \\ \rho \end{array} \right]_{\gamma} \frac{dx_{\mu}}{d\sigma} \frac{dx_{\nu}}{d\sigma} + \sum_{\alpha} \frac{1}{2\lambda} \cdot \frac{d\lambda}{d\sigma} \gamma_{\alpha\rho} \frac{dx_{\alpha}}{d\sigma} - \\ - \sum_{\mu\nu} \frac{1}{2\lambda} \left(\gamma_{\mu\rho} \frac{\partial \lambda}{\partial x_{\nu}} + \gamma_{\nu\rho} \frac{\partial \lambda}{\partial x_{\mu}} - \gamma_{\mu\nu} \frac{\partial \lambda}{\partial x_{\rho}} \right) \frac{dx_{\mu}}{d\sigma} \frac{dx_{\nu}}{d\sigma} = 0. \end{array} \right.$$

Now, if the equation of motion (7) is to be satisfied with $\gamma_{\mu\nu}$ in place of $g_{\mu\nu}$, then the first two terms in (7a) must vanish. Because

$$\frac{d\lambda}{d\sigma} = \sum_{\nu} \frac{\partial \lambda}{\partial x_{\nu}} \frac{dx_{\nu}}{d\sigma},$$

the third term can be put in the form:

$$\sum_{\alpha\nu} \frac{1}{2\lambda} \gamma_{\alpha\rho} \frac{\partial \lambda}{\partial x_{\nu}} \cdot \frac{dx_{\alpha}}{d\sigma} \cdot \frac{dx_{\nu}}{d\sigma} \equiv \sum_{\mu\nu} \frac{1}{2\lambda} \gamma_{\mu\rho} \frac{\partial \lambda}{\partial x_{\nu}} \cdot \frac{dx_{\mu}}{d\sigma} \cdot \frac{dx_{\nu}}{d\sigma}.$$

Since

$$\frac{1}{\lambda} = \frac{g_{\mu\nu}}{\gamma_{\mu\nu}} \neq 0,$$

it thus follows that

$$0 = \sum_{\mu\nu} \left(\gamma_{\nu\rho} \frac{\partial \lambda}{\partial x_{\mu}} - \gamma_{\mu\nu} \frac{\partial \lambda}{\partial x_{\rho}} \right) \frac{dx_{\mu}}{ds} \cdot \frac{dx_{\nu}}{ds}$$

or

$$0 = \sum_{\mu\nu} \left(g_{\nu\rho} \frac{\partial \lambda}{\partial x_{\mu}} - g_{\mu\nu} \frac{\partial \lambda}{\partial x_{\rho}} \right) \frac{dx_{\mu}}{ds} \cdot \frac{dx_{\nu}}{ds}.$$

The directional cosines dx_{μ}/ds have only to satisfy the condition

$$\sum_{\mu,\nu} g_{\mu\nu} \frac{dx_{\mu}}{ds} \frac{dx_{\nu}}{ds} = 1.$$

By means of the same procedure by which equation (6) was derived from (4) and (5), it follows from the last two equations:

$$g_{\nu\rho} \frac{\partial \lambda}{\partial x_{\mu}} - g_{\mu\nu} \frac{\partial \lambda}{\partial x_{\rho}} = \Lambda \cdot g_{\mu\nu}.$$

This holds for arbitrary indices $\mu, \nu, \rho = 1, 2, 3, 4$. If one sets $\rho = \mu$, then it follows that $\Lambda = 0$. For that reason, we have in general

$$g_{\nu\rho} \frac{\partial \lambda}{\partial x_{\mu}} - g_{\mu\nu} \frac{\partial \lambda}{\partial x_{\rho}} = 0 \quad \text{or} \quad g_{\nu\rho}/g_{\nu\mu} = \frac{\partial \lambda}{\partial x_{\rho}} / \frac{\partial \lambda}{\partial x_{\mu}}.$$

$$(\mu, \nu, \rho = 1 \dots 4)$$

Since the terms of two rows of the determinant $|g_{\mu\nu}| \neq 0$ cannot be proportional to one another, all derivatives of λ with respect to the coordinates must therefore vanish. Thus:

$$(8) \quad \lambda = \text{Const.}$$

§12. Consequently, of the reference systems in which whatever constraints imposed on the $g_{\mu\nu}$'s as coordinate functions hold good, the only other reference systems that cannot be distinguished [from them] by observation of the motions of light and uncharged masses are those in which all ratios to one another of the $g_{\mu\nu}$ values of the same or different *coordinate* points have this given [constant] value $[\lambda]$. Likewise, for the derivatives of $g_{\mu\nu}$ with respect to the coordinates, which can be arbitrarily approximated by difference quotients, the satisfaction of whatever conditions are imposed upon them can be verified by observation up to differences which can be obtained by multiplying the values of the $g_{\mu\nu}$'s of all coordinate points by the same constant λ .

In general, there is no coordinate transformation other than the identity which transforms the functions $g_{\mu\nu}$ in the way indicated so that from $g_{\mu\nu} = f_{\mu\nu}(x_1 \dots x_4)$ in Σ we then obtain $g'_{\mu\nu} = \lambda \cdot f_{\mu\nu}(x'_1 \dots x'_4)$ in the transformed system Σ' . This is because any curvature invariant J_σ homogenous in the $g_{\mu\nu}$'s and their differentials (of degree n_σ) must transform from

$$J_\sigma = f_\sigma(x_1 \dots x_4)$$

to

$$J'_\sigma = J_\sigma = \lambda^{n_\sigma} \cdot f_\sigma(x'_1 \dots x'_4).$$

However, the equation following from this,

$$(9) \quad f_\sigma(x_1 \dots x_4) = \lambda^{n_\sigma} \cdot f_\sigma(x_1 \dots x_4),$$

can be satisfied in general only by the identity transformation $x'_1 = x_1 \dots x'_4 = x_4$, since there are more than four independent invariants J_σ homogenous in $g_{\mu\nu}$ and $dg_{\mu\nu}$.²⁴ The only exceptions are cases of peculiarly extensive functional dependence of the $J_\sigma(x_1 \dots x_4)$'s on one another, a thus entirely peculiar state of the invariant curvature state of the space-time manifold.

In general, then, there are no further reference systems that would be equivalent, even from a purely physical point of view, to those that are mathematically distinguished by the constraints on the $g_{\mu\nu}$'s — since those further reference systems would have to belong to the same actual manifold, i.e., would have to be transformable into the latter systems.

²⁴E.g. The eight main components of the curvature tensor, relative to the coordinate directions, which coincide with those of their axes (cf. §§ 14 and 17 in the following), as well as the six angles between these directions.

In the exceptional cases just mentioned, a one-parameter group of transformations, which take $g_{\mu\nu} = f_{\mu\nu}(x_1 \dots x_4)$ into $g'_{\mu\nu} = \lambda \cdot f_{\mu\nu}(x_1 \dots x_4)$, suffices in order to obtain all physically privileged reference systems from the ones mathematically distinguished.²⁵

III. Limiting the Covariance of Einstein's Equations.

§13. Without changing the physical content of Einstein's equations, one can impose on the coordinate functions $g_{\mu\nu}$ occurring in them any condition which can be satisfied in any event solely by a suitable choice of reference system, and thus speak only about this system. Our goal, then, is to select out, by a purely formal transformation of the theory, a narrowest possible group of "privileged" reference systems. The simplest way to achieve this appears to be to connect the coordinate system as closely as possible to the naturally occurring structure of the space-time framework as given, according to Einstein, by its variation in curvature from point to point.²⁶

1. Use of the "Directional Axes" of the Curvature Tensor as Coordinate Directions.

The components $(\lambda\nu, \mu\tau)$ of the Riemann-Christoffel tensor R in $x_1 \dots x_4$ -space with the invariant line element

$$ds = \sqrt{\sum_{\mu\nu} g_{\mu\nu} dx_\mu dx_\nu}$$

are given by²⁷:

$$(10) \quad \left\{ \begin{array}{l} R_{\lambda\nu, \mu\tau} = (\lambda\nu, \mu\tau) = \frac{1}{2} \left(\frac{\partial^2 g_{\lambda\tau}}{\partial x_\nu \partial x_\nu} + \frac{\partial^2 g_{\nu\mu}}{\partial x_\lambda \partial x_\tau} - \frac{\partial^2 g_{\lambda\mu}}{\partial x_\nu \partial x_\tau} - \frac{\partial^2 g_{\nu\tau}}{\partial x_\lambda \partial x_\mu} \right) \\ \quad + \sum_{\alpha\beta} g^{\alpha\beta} \left(\begin{bmatrix} \lambda\tau \\ \alpha \end{bmatrix} \begin{bmatrix} \nu\mu \\ \beta \end{bmatrix} - \begin{bmatrix} \lambda\mu \\ \alpha \end{bmatrix} \begin{bmatrix} \mu\tau \\ \beta \end{bmatrix} \right). \end{array} \right.$$

In this

$$\begin{bmatrix} \alpha\beta \\ \gamma \end{bmatrix} = \frac{1}{2} \left(\frac{\partial g_{\beta\gamma}}{\partial x_\alpha} + \frac{\partial g_{\alpha\gamma}}{\partial x_\beta} - \frac{\partial g_{\alpha\beta}}{\partial x_\gamma} \right)$$

and $g^{\alpha\beta}$ is the normed adjunct of $g_{\alpha\beta}$ in the determinant $|g_{\mu\nu}|$ of $g_{\mu\nu}$. All Greek indices $\alpha, \beta, \lambda, \nu$, etc. range from 1 to 4.

²⁵For the determination of relativity principles actually satisfied in the sense of § 7, these exceptions become insignificant, insofar as they exist only in entirely peculiar types of contingent, i.e. not lawfully determined, physical situations (distribution of matter), which according to Einstein's theory contribute to the determination of the nature of the curvature of the universe, and consequently can be annulled by an (arbitrary) change of these contingencies.

²⁶This possibility of absolutely determining coordinate directions directly by means of the general theory of relativity was pointed out to me in correspondence with G. Mie early in February 1916.

²⁷Christoffel, *Journ. f. Math.* **70**, p. 54. 1869.

The tensor R is covariant of rank 4. Its components thus transform according to the law:

$$(11) \quad (\lambda\nu, \mu\tau)' = \sum_{\alpha\beta\gamma\delta} \frac{\partial x_\alpha}{\partial x_{\lambda'}} \cdot \frac{\partial x_\beta}{\partial x_{\nu'}} \cdot \frac{\partial x_\gamma}{\partial x_{\mu'}} \cdot \frac{\partial x_\delta}{\partial x_{\tau'}} \cdot (\alpha\beta, \gamma\delta).$$

Moreover, the following identities hold:

$$(10a) \quad (\lambda\nu, \mu\tau) \equiv (\mu\tau, \lambda\nu),$$

$$(10b) \quad (\lambda\nu, \mu\tau) \equiv -(\nu\lambda, \mu\tau) \equiv -(\lambda\nu, \tau\mu),$$

thus

$$(\lambda\lambda, \mu\tau) = (\lambda\nu, \mu\mu) = 0$$

and:

$$(10c) \quad (1\ 2, 3\ 4) + (2\ 3, 1\ 4) + (3\ 1, 2\ 4) \equiv 0,$$

so that only 20 of the components of R are algebraically independent of one another. This is because each non-vanishing component is equal to or inversely equal to one of the 36 components $(\lambda\nu, \mu\tau)$, in which $\lambda\nu$ and $\mu\tau$ are equal, respectively, to one of the six numerical pairs 2 3, 3 1, 1 2, 3 4, 2 4, 1 4. Putting the $\lambda\nu$'s in the order of the sequence just written and ordering the $\mu\tau$'s in the same way in columns, the 36 components form the elements of a symmetric matrix, M , whose main diagonal contains the six components of the form $(\lambda\nu, \lambda\nu)$, while the upper half of the auxiliary diagonal is formed from the three left-hand terms of equation (10c). Accordingly, of the 9 distinct diagonal elements of M , which I will call the main components of R , only 8 are algebraically independent. In addition to these, there are the 12 doubly occurring off-diagonal elements of M as independent "auxiliary" components.

§14. One can set these auxiliary components equal to zero at an arbitrary point by an appropriate choice of the directions of the axes of the reference system.

Namely, if $\Sigma'(x_1' \dots x_4')$ is the original reference system with arbitrarily given values of $(\lambda\nu, \mu\tau)'$ and $\Sigma(x_1 \dots x_4)$ the reference system sought in which the 12 auxiliary components of R at the point P in question are to vanish, then this is obtained no matter what if one determines the 16 derivatives

$$\frac{\partial x_\rho}{\partial x_{\sigma'}} \quad (\rho, \sigma = 1 \dots 4)$$

and the 8 independent main components of R at the point P measured in Σ so that the transformation equation (11) is satisfied, while all the auxiliary components of R in Σ appearing on its right-hand side vanish. If one considers, as is

always permissible, the transformation from Σ' to Σ for an infinitesimal neighborhood of P decomposed into a transformation for which the four $\partial x_\rho/\partial x_{\rho'}$ are equal to unity, followed by a pure dilation of the axes for which the 12 remaining

$$\frac{\partial x_\rho}{\partial x_{\sigma'}} \quad (\rho \neq \sigma)$$

vanish, then obviously the last transformation causes, according to equation (11), no new component of R to vanish. The tensor R' , if it can be brought into the desired normal form at all, can be brought into that form also for

$$\frac{\partial x_\rho}{\partial x_{\rho'}} = 1 \quad (\rho = 1 \dots 4).$$

And this is possible, since for the case in which

$$\frac{\partial x_\rho}{\partial x_{\rho'}} = 1$$

and the auxiliary components of R in Σ vanish, the functional determinant calculated from (11) for the 20 independently given components $(\lambda\nu, \mu\tau)'$ does not vanish identically with respect to the twelve

$$\frac{\partial x_\rho}{\partial x_{\sigma'}} \quad (\rho \neq \sigma)$$

for R' in Σ' and the eight independent main components for R in Σ .²⁸ It follows from this at the same time that in general the

$$\frac{\partial x_\rho}{\partial x_{\sigma'}} \quad (\rho \neq \sigma)$$

are definite functions of $(\lambda\mu\nu\tau)'$ [sic], and thus that all auxiliary components of the curvature tensor R vanish only for certain designated directions of the four coordinate axes at a given world point.

²⁸One calculates the functional determinant most easily for [the case]

$$\frac{\partial x_\rho}{\partial x_{\sigma'}} = 0 \quad (\rho \neq \sigma).$$

Of the differentials of the eight main components of R' , then, only those with respect to the corresponding main components of R — appearing in the main diagonal — are different from zero, namely, are equal to one. In the twelve remaining rows corresponding to the auxiliary components of R' we find respectively only three main components resulting from (11) by differentiation with respect to the $\partial x_\rho/\partial x_{\sigma'}$'s, and in fact ordered in such a way that the entire determinant becomes equal to the product of three non-vanishing determinants of the fourth order, whose elements appear in the rows belonging to $(12,13)'$, $(42,43)'$, $(21,24)'$, $(31,34)'$ respectively $(23,24)'$, $(13,14)'$, $(32,31)'$ $(42,41)'$ respectively $(12,14)'$, $(32,34)'$, $(23,21)'$, $(43,41)'$.

§15. Naturally, this system of directions, which, for short, can be called the directional axes of the curvature tensor R , can become underdetermined or in some way degenerate for special cases of R . The most significant special form of R in Einstein's theory is that assumed by R in the "matter free"²⁹ gravitational field. For this field, the Einstein tensor $B_{\mu\nu}$ for the source of the gravitational force³⁰ vanishes, and since, as one easily calculates³¹, $B_{\mu\nu}$ is given by

$$B_{\mu\nu} \equiv \sum_{\lambda\tau} (\lambda\nu, \mu\tau) g^{\lambda\tau},$$

it holds that:

$$(12) \quad B_{\mu\nu} \equiv \sum_{\lambda\tau} (\lambda\nu, \mu\tau) g^{\lambda\tau} = 0.$$

Let it now be possible to choose the coordinate axes and scales at a matter-free world point P so that firstly

$$g^{\mu\nu} = 0 \text{ for } \mu \neq \nu \quad \text{and} \quad g^{\nu\nu} = 1 \quad \left(\begin{matrix} \mu \\ \nu \end{matrix} \right) = 1 \dots 4),$$

or alternatively $g_{\mu\nu} = 0$ ($\mu \neq \nu$) and $g_{\nu\nu} = 1$, which, as is well known, can always be satisfied, and that, secondly, all auxiliary components of R vanish. For the main components of R , there then follow from (12) only the four additional relations:

$$(13) \quad \sum_{\lambda} (\lambda\nu, \lambda\nu) = 0,$$

which reduce the six terms $(\lambda\nu, \lambda\nu)$ of the main diagonal of the matrix, M , of $(\lambda\nu, \mu\tau)$ to two independent ones. Likewise there are only two elements of the auxiliary diagonal independent of one another according to (10c) and (10a), so that one has on the whole exactly four independent main components of R in the reference system chosen.

If one transforms this now by an arbitrary Lorentz transformation, that is, so that it again holds that $g_{\nu\nu} = 1$ and $g_{\mu\nu} = 0$ ($\mu \neq \nu$), then, according to (11), R transforms to a tensor R' which, since the Lorentz transformations have six essential parameters with respect to the components of R ,³² has $4 + 6 = 10$ components independent of one another. Consequently, there can arise among

²⁹A. Einstein, loc. cit. p. 802, 803 § 14. A. Einstein designates "as matter everything except the gravitational field".

³⁰[See the immediately preceding note.]

³¹Cf. A. Einstein and M. Grossmann, loc. cit. p. 35 and 36, eqns. (43), (44) and (46), and A. Einstein, loc. cit. p. 800 and 801, eqns. (43) and (44). The latest version of the Einstein source equations of the gravitational field (A. Einstein, Berl. Ber. p. 142–152. 1917) can unfortunately not be taken into consideration.

³²In order to prove this, one has only to show that there is an infinitesimal Lorentz transformation which transforms R into itself. However, since $\partial x_{\rho} / \partial x_{\rho}' = 1$ up to infinitely small

the 20 components of R' not related by (10a, b, c) no more than $20 - 10 = 10$ independent relations, and these must obviously be the 10 equations which follow entirely generally from (12) for $g^{\nu\nu} = 1$, $g^{\mu\nu} = 0$ ($\mu \neq \nu$).

The possibility mentioned above, of choosing the coordinate directions at a matter-free world point, $B_{\mu\nu} = 0$, so that they coincide with the directional axes of R and $g_{\mu\nu} = 0$ as well for $\mu \neq \nu$, involves no restriction beyond (12) for R , and thus exists in general. If, as is customary, one takes coordinate directions perpendicular to one another³³ for which $g_{\mu\nu} = 0$ ($\mu \neq \nu$), then in consequence, at each matter-free world point, the four directional axes of the world curvature tensor stand perpendicular to one another. The presence of “matter” in general causes it to be the case that the coordinate system of R becomes obliquely angled.

§16. There does not exist in general a reference system whose directional axes everywhere coincide with those of the curvature tensor R because of the integrability condition necessary for it. In contrast, one can of course require that the x_4 -direction ($dx_1 = dx_2 = dx_3 = 0$) coincides everywhere with one of these directions.³⁴ If one thinks of this requirement expressed as a condition on the coordinate functions $g_{\mu\nu}$, which determine the curvature tensor R according to (10), and introduces it into Einstein’s equations, then the resulting system of equations is not invariant in each case under continuous rotations about the x_4 -axis, i.e., [under] velocity transformations proper. The reference systems thus picked out mathematically are in general, according to § 12, also distinguished from all others observationally.

In the sense of the interpretation presented in Part I, Einstein’s theory thus satisfies no relativity principle of velocity.³⁵

terms of the second order in virtue of [the fact that] in such a case

$$\sum_{\sigma} (\partial x_{\rho} / \partial x_{\sigma'})^2 = 1,$$

the claim already follows, that the functional determinant of the 12 auxiliary components with respect to the 12 infinitely small transformation coefficients $\partial x_{\rho} / \partial x_{\sigma'}$ ($\rho \neq \sigma$), formed according to (11), is different from zero (cf. the first footnote of § 14; equation (13) changes nothing in this), and thus these coefficients can be expressed by the components of R and R' and thereby can have a null value in agreement with (11) only for $R' = R$.

³³This follows directly, if one defines the cosines of the angles α between two directions:

$$(dx_1/ds)_1, \dots, (dx_4/ds)_1 \text{ and } (dx_1/ds)_2, \dots, (dx_4/ds)_2$$

by:

$$\cos \alpha = \sum_{\mu\nu} g_{\mu\nu} (dx_{\mu}/ds)_1 \cdot (dx_{\nu}/ds)_2.$$

Cf., e.g., L. Bianchi-Lukat: “Vorlesungen über Differentialgeometrie”.

³⁴Besides this one can, e.g., also require that for $x_4 = 0$ the x_3 -direction, for $x_4 = x_3 = 0$ the x_2 direction, and finally the x_1 axis, $x_4 = x_3 = x_2 = 0$, coincides everywhere with one of the directional axes.

³⁵On the other hand, with respect to the introduction of the reference system designated in the way indicated, the equations still remain invariant under the infinite group of transfor-

2. On the Introduction of Absolute Invariants as Space and Time Coordinates.

§17. In many more definite and, at the same time, simpler ways one can nevertheless specify reference systems by means of the absolute invariants which can be formed from the components of R and $g_{\mu\nu}$. Since only 16 coefficients, $\partial x_\nu / \partial x_\rho'$, enter into the transformation laws of the tensor components $(\lambda\nu, \mu\tau)$ and $g_{\mu\nu}$, among which altogether $20 + 10 = 30$ are independent, so in general there must be $30 - 16 = 14$ such algebraically independent invariants, and in matter-free regions, where the 10 equations (12) hold, still $14 - 10 = 4$. In this case, the invariants obviously must be functions of the (invariant) main components of R measured in the orthogonal infinitesimal reference system $\Sigma(dx_1 \dots dx_4)$, whose directional axes coincide with those of R and for which $g_{\nu\nu} = 1$. As shown in the general case, $B_{\mu\nu} \neq 0$, the number of independent components of R in an infinitesimal reference system, determined in a corresponding way according to position and scale, increases to 8, and there are in addition six quantities which measure the angles between the directional axes of R . Altogether this yields exactly 14 independent absolutely invariant functions of $(\lambda\nu, \mu\tau)$ and $g_{\mu\nu}$.

Now, if J_1, J_2, J_3, J_4 are four functions of these invariants which are independent of one another for the case $B_{\mu\nu} \neq 0$ as for $B_{\mu\nu} = 0$, then a reference system is completely specified mathematically and, according to § 12, is physically distinguished from all the rest, if one sets:

$$(13) \quad x_1 = J_1, \quad x_2 = J_2, \quad x_3 = J_3, \quad x_4 = J_4.$$

§18. The introduction of one of the reference systems determined by equations of the type (13) into the general theory of relativity, which would thereby mathematically be brought into the form of a completely "absolute" theory, is unobjectionable, however, only under the assumption that the invariants $J_1 \dots J_4$ chosen as coordinates are functionally dependent on one another in no finite four-dimensional region; for otherwise, whole continua of world points would be designated by each coordinate point.

The question now arises, whether this assumption, essential for the introduction of the reference systems (13), which, because of the invariance of $J_1 \dots J_4$, undoubtedly has physical content, adds anything physically new to Einstein's theory. For only if this is not the case can the proof, as given above — that Einstein's theory is physically a completely absolute one according to the interpretation of physical relativity principles expounded in Part I — be valid.

Assuming the correctness of Einstein's theory, then in the actual, observable world any functional dependence of $J_1 \dots J_4$ on one another in a region extending finitely in space and time can be deemed infinitely improbably or impossible, since it constitutes only a single case among infinitely many cases of complete independence of $J_1 \dots J_4$ on one another.

mations which leaves the x_4 axis everywhere unchanged.

However, the laws of Einstein's theory, just as those any other physical theory, determine by and for themselves not so much what actually happens, but what is "possible" at all, i.e., is consistent with them. The assumption of the independence of $J_1 \dots J_4$, however, excludes certain clear, albeit singular possibilities that are open according to them; and in fact, because of the invariance of the J_ν 's, these are also topologically distinguished from all the rest, and consequently, distinguishable in principle from them by observation. In this sense, the assumption in question thus constitutes a physical supplement to Einstein's theory.

3. Closer Determination of Reference Systems by Conditions Imposed on $g_{\mu\nu}$.

§19. For this reason, we should look for another way of completely removing from Einstein's theory those of its covariance properties which are merely formal, a way which is not subject to the worry just voiced. The technique consists simply in this: the coefficients $g_{\mu\nu}$ in the expression of the world line element, which occur in the general theory of relativity as partially undetermined coordinate functions subject to free choice, are to be determined more precisely by [imposing] as many arbitrary conditions as can be satisfied, without new physical assumptions, solely by a suitable choice of reference system, which in that way becomes exactly determined.

Einstein himself introduced such a condition, namely $|g_{\mu\nu}| = 1$, in order to simplify his equations,³⁶ and, coincidentally, noted further that corresponding to the four arbitrarily selectable coordinates, four functions of $g_{\mu\nu}$ as coordinate functions in general can be chosen freely.³⁷

If one stipulates, e.g.,

$$(14) \quad g_{11} = g_{22} = g_{33} = g_{44} \equiv 1 \text{ for all } x_1, x_2, x_3, x_4,$$

then in order to transform a given reference system $\Sigma'(x_1', x_2', x_3', x_4')$ into a system $\Sigma(x_1, x_2, x_3, x_4)$ in which (14) holds, one has only to satisfy the four differential equations

$$(15) \quad 1 = \sum_{\mu\nu} g_{\mu\nu}'(\varphi_1, \dots, \varphi_4) \frac{\partial \varphi_\mu}{\partial x_\alpha} \cdot \frac{\partial \varphi_\nu}{\partial x_\alpha} \quad (\alpha = 1 \dots 4)$$

by a suitable choice of the four functions $\varphi_\nu(x_1 \dots x_4) = x_\nu'$ or their inverse functions $f_\nu(x_1' \dots x_4') = x_\nu$, which take Σ' to Σ . This is always possible.³⁸

³⁶A. Einstein. loc. cit. p. 801. According to Mr. Einstein's interpretation, there lies in this however no proof that his theory (in other form) cannot satisfy the most general relativity principle.

³⁷A. Einstein, loc. cit. p. 812 above and note 1.

³⁸To begin with, one can consider Σ' transformed to a(n) (equivalently characterized) reference system whose directional axes coincide nowhere with the direction $ds = 0$ either in the past or future surface of the light cone. Then $g_{\nu\nu}' \neq 0$ ($\nu = 1 \dots 4$) everywhere, and equation (15) can be solved for the four fully independent differential quotients $\partial \varphi_\nu / \partial x_\nu$ ($\nu = 1 \dots 4$).

What's more, the functions φ_ν can be chosen arbitrarily in a space $x_\nu = \text{Const.}$, for instance $x_\nu = 0$. (Compare note 1.) Besides (14) then, a further condition on φ_1 , [viz.],

$$g_{12} = 0 \text{ for } x_1 = 0,$$

may be imposed. If one considers — in rough approximation — the differential equations (further) arising from this for φ_ν solved in regard to one of the (as yet undetermined) derivatives

$$\frac{\partial \varphi_1}{\partial x_2}, \frac{\partial \varphi_1}{\partial x_3}, \frac{\partial \varphi_1}{\partial x_4}$$

of φ_1 , say in regard to $\partial \varphi_1 / \partial x_2$, then it follows that for the determination of φ_1 we may also require

$$g_{13} = 0 \text{ for } x_1 = x_2 = 0$$

and, finally, also at the same time

$$g_{14} = 0 \text{ for } x_1 = x_2 = x_3 = 0.$$

Corresponding requirements for the determination of the remaining transformation functions $\varphi_2, \varphi_3, \varphi_4$, result from those for φ_1 by cyclic permutation. On the whole, one obtains in addition to (14) the conditions:

$$(16) \quad \left\{ \begin{array}{llll} g_{12} = 0 & \text{for } x_1 = 0 & \text{and for } x_2 = x_3 = x_4 = 0 \\ g_{23} = 0 & \text{" } x_2 = 0 & \text{" } x_3 = x_4 = x_1 = 0 \\ g_{34} = 0 & \text{" } x_3 = 0 & \text{" } x_4 = x_1 = x_2 = 0 \\ g_{41} = 0 & \text{" } x_4 = 0 & \text{" } x_1 = x_2 = x_3 = 0 \\ g_{24} = 0 & \text{for } x_2 = x_3 = 0 & \text{and for } x_4 = x_1 = 0 \\ g_{31} = 0 & \text{" } x_3 = x_4 = 0 & \text{" } x_1 = x_2 = 0. \end{array} \right.$$

The conditions (14) and (16) are obviously chosen so that the "natural" measurement system, $g_{\nu\nu} = 1$, $g_{\mu\nu} = 0$ ($\mu \neq \nu$), holds at the origin 0 ($x_1 = x_2 = x_3 = x_4 = 0$) and, as far as is possible, in its neighborhood and on the coordinate axes, coordinate surfaces, and coordinate spaces.

According to the provisional consideration set out above, there remain undetermined only: first, the values of $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ at 0, and second, yet six more parameters codetermining the functions φ_ν , say the values of six derivatives of φ_ν at 0, since naturally among the conditions determining φ_ν the vanishing of each of the six quantities $g_{\mu\nu}$ ($\mu \neq \nu$) at 0 is counted twice. Accordingly, with a given form of the invariant world curvature, there is in general for each world point as origin an exact six-parameter class of reference systems in which equations (14) and (16) are mathematically satisfied.

§20. The rigorous proof for this could be obtained only for the neighborhood of 0 under the assumption that the $g_{\mu\nu}$, $g_{\mu\nu}'$ and φ_ν are analytic functions of their arguments.

In this case, the conditions (14) and (16) are completely satisfied if only all the derivatives of $g_{\mu\nu}$ at 0 equal 0, whose vanishing at 0 follows from (14) and (16), and each function $\varphi_\nu(x_1 \dots x_4)$ is completely determined if its own value and the values of all its derivatives at 0 are determined.

In order to investigate now the extent to which a reference system $\Sigma(x_1 \dots x_4)$ is defined by the conditions (14) and (16), I assume that the origin of the reference system $\Sigma(x_1 \dots x_4)$ coincides with that of a given fixed reference system $\Sigma'(x_1' \dots x_4')$ and that the equations (14) and (16) are satisfied equally in Σ for $g_{\mu\nu}(x_1 \dots x_4)$ as in Σ' for $g_{\mu\nu}'(x_1' \dots x_4')$. Then to begin with, for the 16 first derivatives of

$$x_1' = \varphi_1(x_1 \dots x_4), \dots, x_4' = \varphi_4(x_1 \dots x_4)$$

there follow from

$$(17) \quad g_{\alpha\beta} = \sum_{\mu\nu} \frac{\partial\varphi_\mu}{\partial x_\alpha} \cdot \frac{\partial\varphi_\nu}{\partial x_\beta} \cdot g_{\mu\nu}'$$

according to (14) and (16), the 10 equations:

$$(18a) \quad g_{\alpha\alpha} = 1 = \sum_{\nu} \left(\frac{\partial\varphi_\nu}{\partial x_\alpha} \right)^2,$$

$$(18b) \quad g_{\alpha\beta} = 0 = \sum_{\nu} \frac{\partial\varphi_\nu}{\partial x_\alpha} \cdot \frac{\partial\varphi_\nu}{\partial x_\beta} \quad (\alpha \neq \beta).$$

The last six equations say that in Σ as well as in Σ' the initial directions of the coordinate axes stand perpendicular to one another. If one lets them coincide, so that

$$\frac{\partial\varphi_\nu}{\partial x_\alpha} = 0 \quad \text{for } \nu \neq \alpha$$

and, following (18a)

$$\frac{\partial\varphi_\alpha}{\partial x_\alpha} = +1,$$

then six further parameters of determination for Σ (besides the signs of the $\partial\varphi_\alpha/\partial x_\alpha$'s) are freely chosen by this.

Since according to (14) and (16), all first derivatives of $g_{\mu\nu}(x_1 \dots x_4)$ and $g_{\mu\nu}'(x_1' \dots x_4')$ vanish at 0, by differentiation of (17) one obtains for the 40 second derivatives of $\varphi_\nu(x_1 \dots x_4)$ the 40 equations

$$0 = \frac{\partial g_{\alpha\beta}}{\partial x_\sigma} \equiv \sum_{\mu\nu} g_{\mu\nu}' \left(\frac{\partial^2\varphi_\mu}{\partial x_\alpha \partial x_\sigma} \cdot \frac{\partial\varphi_\nu}{\partial x_\beta} + \frac{\partial^2\varphi_\nu}{\partial x_\beta \partial x_\sigma} \cdot \frac{\partial\varphi_\mu}{\partial x_\alpha} \right),$$

or by considering the values established for the $g_{\mu\nu}$'s, $g_{\mu\nu}$'s and the first derivatives of φ_ν :

$$0 = \frac{\partial^2\varphi_\beta}{\partial x_\alpha \partial x_\sigma} + \frac{\partial^2\varphi_\alpha}{\partial x_\beta \partial x_\sigma} \quad (\alpha, \beta, \sigma = 1 \dots 4).$$

The equations for the 3rd derivatives of φ_ν and for each higher order result in a corresponding fashion, in which one forms by means of (17) the expressions for all derivatives of $g_{\alpha\beta}$ one degree lower which vanish according to (14) and (16), and sets them equal to zero. One thereby obtains for the

$$4 \cdot \frac{4 \cdot 5 \cdot 6 \dots (4+n)}{(n+1)!}$$

derivatives of the $(n+1)$ th order of φ_ν , according to (14) and (16), also exactly as many³⁹ equations.

If all derivatives of φ_ν of the 2nd up to the n th order inclusively vanish, then these equations have the simple form⁴⁰

$$(19) \quad 0 = \frac{\partial^n g_{\alpha\beta}}{\partial x_{\sigma_1} \cdot \partial x_{\sigma_2} \dots \partial x_{\sigma_n}} \equiv \frac{\partial^{n+1} \varphi_\beta}{\partial x_\alpha \cdot \partial x_{\sigma_1} \dots \partial x_{\sigma_n}} + \frac{\partial^{n+1} \varphi_\alpha}{\partial x_\beta \cdot \partial x_{\sigma_1} \dots \partial x_{\sigma_n}}$$

$(\alpha, \beta, \sigma_1, \dots, \sigma_n = 1 \dots 4).$

The derivatives of φ_ν thus vanish from the 2nd up to any arbitrary order if only the discriminant D_n of all corresponding determination equations (19) differ from 0.

The investigation of these discriminants is not difficult, although somewhat protracted. It shows that they satisfy the restriction in question, with the exception of the discriminant formed for $n = 2$, corresponding to the second derivative of the $g_{\mu\nu}$, which alone vanishes.⁴¹

³⁹The first 4 equations (16) supply $1 + \frac{3 \cdot 4 \dots (n+2)}{n!}$ equations each and the last 2 yield $2 + \frac{2 \cdot 3 \dots (n+1)}{n!}$ equations. Altogether with the $4 \cdot \frac{4 \cdot 5 \dots (n+3)}{n!}$ equations this amounts to:

$$4 \cdot \frac{4 \cdot 5 \dots (n+4)}{(n+1)!} \times$$

$$\frac{1 \dots 2 \cdot 3 + 3(n+1) \cdot (n+2) + 2 \cdot 3 \cdot (n+1) + (n+1) \cdot (n+2) \cdot (n+3)}{(n+2) \cdot (n+3) \cdot (n+4)}$$

$$\equiv 4 \cdot \frac{4 \cdot 5 \dots (n+4)}{(n+1)!},$$

as claimed.

⁴⁰Because

$$\frac{\partial x'_\nu}{\partial x_\sigma} \equiv \frac{\partial \varphi_\nu}{\partial x_\sigma} = 0 \text{ for } \nu \neq \sigma$$

on the right hand side occur only the same differential quotients of $g_{\mu\nu}$ with respect to $x'_1 \dots x_4$ [sic] as of $g_{\mu\nu}$ with respect to $x_1 \dots x_4$ on the left, and thus simply vanish.

⁴¹It follows from (14), (16), and (19) that, just as in each row of D_n , also in each column only one or two elements can differ from zero, and thus can be equal to one. If one imagines each pair of nonvanishing elements of the same row or column joined by a straight line, then there emerge curves which can be closed only for $n = 1$ and $n = 2$, and, if they are open, always join an odd number of elements. Each discriminant D_n forms a product of the respective subdeterminants formed from the elements of each curve. Of these none vanish except those

If instead one sets equal to zero another suitably chosen 2nd derivative of $g_{\mu\nu}$ (in Σ and in Σ') not determined by (16), e.g.,

$$(20) \quad \frac{\partial^2 g_{24}}{\partial x_1 \cdot \partial x_3} = 0$$

and, instead of the equation derived from

$$(21) \quad \frac{\partial^2 g_{12}}{\partial x_3 \cdot \partial x_4} = 0,$$

writes the equation that follows according to (17), then instead of D_2 there appears a new discriminant, D_2^* , which proves to be different from 0.⁴² With the help of the new constraint (20), all of the resulting derivatives of φ_ν are forced to vanish, except those of the four $\partial\varphi_\nu/\partial x_\nu$, which are equal to 1. The reference system Σ coincides with Σ' and is thereby completely fixed.

In doing so, the just excluded condition (21) is not yet used for its determination. If we assume the remaining equations following from (14) and (16) for the first and second derivatives of $g_{\mu\nu}$, then, according to (10), it can be replaced by the equation

$$(21a) \quad R_{2413} \equiv (24, 13) = 0,$$

which, in conformity with the transformation equations (11) for $R_{\lambda\nu\mu\tau}$, is always satisfiable by a suitable rotation at 0 of the directional axes of Σ perpendicular to one another. The condition (21) can thereby in general⁴³ replace one of the six parameter constraints which should determine these directions.

It needs to be proved further that there is, for each form of the invariant world curvature, at least one reference system in which the conditions (14),

of the closed curves — for $n = 2$ uniquely — which join the four rows belonging to

$$\frac{\partial^2 g_{12}}{\partial x_3 \partial x_4}, \frac{\partial^2 g_{23}}{\partial x_4 \partial x_1}, \frac{\partial^2 g_{34}}{\partial x_1 \partial x_2}, \frac{\partial^2 g_{41}}{\partial x_2 \partial x_3}.$$

Thus, only $D_2 = 0$. If one replaces one of the differential quotients just computed by

$$\frac{\partial^2 g_{24}}{\partial x_1 \partial x_3} \text{ or } \frac{\partial^2 g_{13}}{\partial x_4 \partial x_2},$$

then the correspondingly formed subdeterminant assumes the value $+2$. $D_2^* \neq 0$. This is connected with [the fact] that in expression (10) for the components of the curvature tensor R the last two mentioned differential quotients appear — in $R_{12\ 34}$, $R_{23\ 14}$, $R_{31\ 24}$ — only summed and combined with no other second derivatives of the $g_{\mu\nu}$ as the four previously mentioned.

⁴²Cf. the immediately preceding footnote.

⁴³The sole exception occurs in the case in which it holds that $R_{24\ 13} = 0$ at 0 for each position of the orthogonal directional axes of Σ . As one can infer from the transformation laws (11) for $R_{\lambda\nu\ \mu\tau}$, this occurs only if [the origin] 0 lies at a world-point at which all components of the curvature tensor R vanish. However, this case can be excluded by the condition $R \neq 0$ at 0, which, according to the general theory of relativity, is always satisfiable except in an entirely matter and gravitation free, i.e., entirely empty universe ($R \equiv 0$).

§22. It is to be noted further that the transformations between the reference systems picked out by any further conditions on $g_{\mu\nu}$ for the various possible invariant curvature conditions⁴⁵ of the world also form, in general, different groups.⁴⁶ According to § 7, however, Einstein's theory can satisfy, in a physical sense, only a relativity postulate whose group is admitted [berechtigt] independently of the contingent metrical character of actuality, i.e., equal or similar to a subgroup of any one of the groups mentioned.⁴⁷ The widest group which simultaneously satisfies this condition for each physically adequate form of presentation of the theory, determines the relativity principle actually satisfied by the theory.

IV. Geometrical Determination of the Relativity Principle Satisfied by Einstein's New Theory of Relativity and Comparison with the Original Theory of Relativity

§23. Which group this is, or: which relativity principle in the sense of § 7 Einstein's theory satisfies, i.e., independently of the form of presentation of the theory and the contingent physical conditions of actuality, one can recognize more simply and clearly if, rather than seeking to multiply the conditions imposed on $g_{\mu\nu}$ up to the limit of permissibility, one considers the four-dimensional geometric picture which describes the law of motion, $\delta \int ds = 0$, of Einstein's theory for light rays, ($ds = 0$), and point masses,

$$(ds^2 = \sum g_{\mu\nu} dx_\mu dx_\nu < 0),$$

in an arbitrary coordinate manifold $\Sigma(x_1 \dots x_4)$. For each system of arbitrarily given coordinate functions $g_{\mu\nu} = f_{\mu\nu}(x_1 \dots x_4)$, ($|g_{\mu\nu}| > 0$), the equation

$$(22) \quad \delta \int ds = 0$$

with $ds^2 \leq 0$ determines an infinite class of world lines, the extremal curves of the manifold $\Sigma(x_1 \dots x_4)$. Each of the world lines represents a theoretically

⁴⁵It would be pointless to demand that, in one and the same reference system privileged with respect to a given form of Einstein's theory, this form must be valid at the same time for two different or even for all possible curvature conditions of the universe. For at no world-point, and hence, also at no coordinate point of any admissible reference system whatsoever can two different values of the same curvature invariant be proposed as simultaneous valid according to the theory.

⁴⁶Excepted is the case in which it is stipulated that only $|g_{\mu\nu}| = \text{Const.}$ Here the distinguished reference systems are always associated with the same group of transformations, determined by $|g_{\mu\nu}| / |g'_{\mu\nu}| = \text{Const.} = |\partial x_\mu| / |\partial x'_\nu|^2$.

⁴⁷Such a group is the group of transformations, with reference to which the equations put forward are covariant. But this is not to say that it is the widest group of the sort required, since only similarity [isomorphism — Ähnlichkeit], not sameness of form [literally sameness? — Formgleichheit] is required for a subgroup of each of the various groups picked out under various physical circumstances.

possible motion of a point mass ($ds < 0$) or light ray in the aether, and their totality [represents] the totality of the motions of light rays and point masses possible in the same space-time manifold.⁴⁸ The set [Menge] of all these classes of world lines associated with the various function systems $g_{\mu\nu} = f_{\mu\nu}(x_1 \dots x_4)$ divide into infinitely many subsets [Teilmengen], each [one] of which contains all [the] classes of world lines which are topologically equivalent and hence cannot be distinguished from one another purely by observations of motions of light and masses — except by [invoking an] arbitrary relation to some particular reference system. Corresponding to each extremal class given by some particular system of functions $g_{\mu\nu} = f_{\mu\nu}(x_1 \dots x_4)$ is an infinity of others, according to Part II, which are not only topologically equivalent to it, but which fully coincide with it, namely all those which correspond to systems of functions of the form $g_{\mu\nu} = \lambda \cdot f_{\mu\nu}(x_1 \dots x_4)$, where λ is independent of $x_1 \dots x_4$. Each of the subsets [Teilmengen] in question can thus be divided into an infinity of constituent sets [Untermengen — suggestions welcome here], each of which contains, metrically, exactly the same class of world lines. Those differences in the invariant curvature condition of the space-time manifold which are due to the variation of the value of the parameter λ associated with each component subset according to § 12 do not enter physically into the expression of the laws of motion of Einstein's theory for light rays and point masses.

§24. The absolute mathematical invariance of Einstein's laws of motion requires and is obviously due to the fact that each of the constituent sets [Untermenge] in question contains all classes of world lines which can arise from one another by a continuous [stetige] deformation; for according to what has been said above, it is impossible to map a class of world lines from one constituent set [Untermenge] to another with a continuous [stetige] transformation.

The physical content, [i.e., the content] verifiable by purely (topological) observations, of what is depicted by the classes of world lines of a constituent set [Untermenge] is already fully given, however, by any single arbitrarily chosen class of the set [Menge], since, obviously, there does not exist a topological difference between the classes of the same set [Menge], and any class can be converted into any other simply by an appropriate choice of reference system. All classes of world lines except for an arbitrarily chosen one may be eliminated from each constituent set [Untermenge] without thereby restricting or otherwise altering the physical content of the description vis-à-vis Einstein's equation (22). On the contrary, in this process only such parts of the picture given by (22) which signify nothing essential would be removed, being redundant representations of an already represented physical possibility, although they are, as mentioned, indispensable for achieving the mathematical invariance of the theory and probably also its mathematically elegant form. But on the other hand, a representation

⁴⁸Possible means always: in conformity with the laws assumed, and so here: in agreement with the laws of motion cited. Cf. the third footnote in § 7.

of the sort described above⁴⁹ satisfies the demand of utmost conciseness.⁵⁰

§25. The actual reference systems in which such a description is satisfied, i.e., in which the world lines of the actual light rays and point masses belong to one of the extremal classes satisfying that description [lit. – contained in it], can, because of the purely kinematical character of what is described, be distinguished in principle from all the others by suitable observations (cf. § 4). Consequently, Einstein's theory physically satisfies no relativity principle for which not every single one of the extremal classes in question is self-invariant [für sich invariant] with respect to the invariance group of transformations corresponding [to that relativity principle]; for inasmuch as the classes are altogether topologically different or [else] are distinguished according to § 12 by the generally constant parameter λ , a transformation from the one into the other is in general impossible. An entire extremal class of the space-time manifold can be mapped onto itself, according to Part II, only with a transformation which leaves the corresponding system of functions $g_{\mu\nu} = f_{\mu\nu}(x_1 \dots x_4)$ unchanged up to a constant factor λ , and, except for the identity, there is no coordinate transformation which in general allows this. With regard to any other transformation, invariance exists only in cases that are not only exceptional [singulären], but in actuality are no doubt out of question (cf. § 12).

Consequently, Einstein's theory physically satisfies no relativity principle whatsoever in the sense developed in § 7; it is a completely absolute theory in regard to its content.

§26. A theory, which admits the law of motion (22) for light rays and point masses, or otherwise assigns a physical significance to the extremal curves described, can satisfy relativity principles in the sense of § 7 only if at the same time it permits as possible only those exceptional forms of the invariant world curvature in which the functions $g_{\mu\nu} = f_{\mu\nu}(x_1 \dots x_4)$ can assume the form

$$g_{\mu\nu}' = \lambda \cdot f_{\mu\nu}(x_1' \dots x_4')$$

for a coordinate transformation different from the identity and consequently the relevant extremal class can be transformed into itself. The more a theory restricts in this way the possibility of different sorts of world curvature

⁴⁹Analytically one arrives at such a representation by the procedure used in Part III, 3 of imposing conditions on the $g_{\mu\nu}$ as coordinate functions which can be satisfied only by a suitable choice of the reference system independently of the contingent invariant metrical character of the space-time manifold and which consequently can never exclude any collections of geodesics [Extremalenscharen] transformable into one and proportionally related to it by λ . Whether it can be fully obtained in this way, which in any case proceeds straightforwardly [recht weit führt], as shown in §§ 20, 21, is questionable. However, the main point is not to produce a representation of the characteristic sort in mathematically closed form, but it suffices that it is conceivable at all, if only as a geometrical picture.

⁵⁰It is assumed that the parameter λ has an actual physical meaning. Otherwise, from the infinity of topologically equivalent sets-of-world-lines contained in the representation and distinguished only by the designated value of λ , all but one would have to be eliminated.

conditions,⁵¹ the more far reaching a relativity postulate can it satisfy. In any event, the extreme limit is achieved if, as happens in Einstein's original theory of relativity, the world curvature tensor is set identically equal to zero. In this case, as is well known, there is only one extremal class, and this is invariant with respect to the Lorentz group composed with the group of uniform dilations and translations, i.e., it goes over into itself under each rotation, displacement and uniform expansion of the space-time manifold.

It thus follows that: *A physical theory, which confers on the extremal curves (22) of a space-time manifold, with line elements of Minkowski normal form, a significance accessible to observation, or represents in some other way the invariant metrical character of the manifold as observable in principle to the same extent, can satisfy no wider a relativity principle in the sense of § 7, than that of Einstein's original theory of relativity.* In light of the interpretation of the meaning of physical relativity principles defended here, this theory thus appears not as an intermediate among equally possible relativity theories, some more special, some more general, but as the fulfillment of the widest relativity principle which can be satisfied at all under the already mentioned assumption that the invariant world curvature condition determined by the extremal world lines be in some way observable in principle. The opposite extreme is realized in Einstein's new theory, which, as shown above, physically satisfies no relativity principle whatsoever.

Conclusion:

On the Reason for the Unsatisfiability of the General Principle of Relativity

§27. Just as with the laws of motion considered, so also other kinematical laws determine, solely in virtue of their topological content verifiable by observation, an invariant transformation group unique to them, whose relativity principle they satisfy.

If, e.g., one puts the kinematical laws of rigid bodies, or more exactly speaking, the kinematical laws of rigidly joined point masses, in the form:⁵²

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 = r^2 = \text{Const.}$$

$$t_1 - t_2 = 0,$$

then the group of transformations which carry into themselves the infinite class of world lines and systems of world lines described by this equation, as an invariance group — not distinguished from any other up to isomorphism [von allen ihr ähnlichen] — is determined only by the topology of this class of world lines, and since the equations express the laws contained in them without redundancy,

⁵¹This can be done by imposing invariant conditions on $g_{\mu\nu}$, the most exceptional of which, $R_{\lambda\nu,\mu\tau} \equiv 0$, requires the identical vanishing of the world curvature.

⁵²In this form, the impenetrability of rigid bodies is not taken into account.

each representation of the same laws must thus satisfy the relativity principle belonging to this group. Thus, there exists in each case a physical invariance with respect to the group of all transformations, formed variously according to the choice of reference system, which leave invariant the spaces of constant time and the spatial distances of simultaneous world points. Any further transformation is physically inadmissible [unberechtigt], regardless as to whether or not the equations are put in a form invariant with respect to it.

§28. This raises the question, which general properties of kinematical laws ground the selection of some particular invariance group of coordinate transformations as the only physically acceptable one, and whether kinematical laws cannot be conceived which also physically satisfy the most general principle of relativity.

Suppose such a system of laws is given. If one again imagines that its content is described by a class, or more generally, (infinitely) many altogether topologically distinct classes of world lines, surfaces, etc., each of which reproduces a totality of motions and configurations possible in the same space-time manifold according to the system of laws, then each individual class must be carried onto itself again under each arbitrary continuous deformation. If one thus picks out from any class an arbitrary part consisting of some system of world lines or such like, then this part is contained not only in the form and place selected in the class, but at the same time in all other arbitrarily distorted and shifted forms and places which are contiguous with or interpenetrate the first in any way conceivable. But this means: a system of laws physically satisfying the general principle of relativity can exclude, among the spatio-temporal situations for which it is valid, no types of mutual contiguity or interpenetration which are at all geometrically conceivable in the preservation of the topological character of the individual situation.

On the other hand, such a system of laws may well require the (regular) *unconditional*⁵³ existence of certain contiguities or interpenetrations. If one imagines these once given in the geometrical picture of the laws, it then follows from the absolute invariance of the picture alone that they must be found at the same time in every other place in [the picture] that results from [a] displacement or distortion. But this cannot in itself ever exclude or restrict their presence.

Consequently, it is thus the negative content of the kinematical laws known to us, i. e., the restricting and denying of possibilities of coincidence, which makes impossible the physical satisfaction of the general principle of relativity.

§29. With the law considered above for the motion of light and matter in the aether, which identifies the world lines of light rays and point masses with the extremal curves of the space-time manifold, the negative part of the content apparently consists in the proposition following from this equation [? — Gleichsetzung], that through two distinct world points there can never go

⁵³If the contiguities or interpenetrations were only necessary consequences of other coincidences, then the existence of these latter would themselves alone be out of the question, and as a result the laws could not satisfy the general principle of relativity.

two of the distinct world lines mentioned. In fact, it is precisely this proposition which limits in each coordinate manifold the class of mutually consistent world lines and thereby as well the group of transformations which takes it into itself — which in general is only the identity transformation — since it would be violated by adding any further world line to the class.

Likewise, systems of rigidly connected point masses are usable in principle for measurement (e.g., as scale points of measuring rods) and thereby for the physical designation of a certain transformation group only in virtue of the exclusion of free motion of the point masses with respect to one another.

As an example of an unconditional, purely affirmative kinematical law, one could perhaps advance the proposition that in the course of time each point mass collides with at least one other and then must remain joined with it forever after. The geometrical picture which contains the totality of world lines not contradicting this fictitious law must map onto itself under each continuous coordinate transformation, for its overlap with an arbitrarily distorted picture can never yield a singly standing world line running together with no other, and thus no world line not already contained in the original picture.

In fact, however, kinematical laws of this purely affirmative sort have never been proposed. *But only then, if exclusively such unconditional and affirmative laws were to govern the world instead of the only known negative and conditional ones, would the general principle of relativity have objective validity in the sense of § 7.*