More on Models of Theories

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Definitions Related to Theories

Defn. T is a theory iff $T = \text{Cn } T$.

Defn. Let $Sent(\mathcal{L})$ be the set of sentences of a given elementary language $\mathcal{L}$ and let $\mathcal{K}$ be a class of structures for $\mathcal{L}$. Then

$$\text{Th } \mathcal{K} =_{df} \{\sigma \in Sent(\mathcal{L}) \ : \ \models_{\mathfrak{A}} \sigma \text{ for each } \mathfrak{A} \in \mathcal{K}\}.$$

Lemma. Th $\mathcal{K}$ really is a theory (for any $\mathcal{K}$).

Proof. Exercise.

Defn. Let $\Sigma \subset Sent(\mathcal{L})$. Then Mod Σ is the class of models of Σ, i.e.,

$$\text{Mod } \Sigma =_{df} \{\mathfrak{A} \in Str(\mathcal{L}) \ : \ \models_{\mathfrak{A}} \tau \text{ for each } \tau \in \Sigma\}.$$

More Definitions

Defn. Σ is **decidable** iff Σ is decidable as a subset of $Sent(\mathcal{L})$.

Defn. A theory T is **decidable** iff T is decidable as a set of sentences.

Defn. Theory T is **axiomatizable** iff there is a decidable set Σ of sentences s.t. $\text{Cn } \Sigma = T$.

Defn. T is **finitely axiomatizable** (f.a.) iff there is a sentence τ s.t. $\text{Cn } \tau = T$.

Defn. T is complete iff ,for every sentence σ of $\mathcal{L}$, $\sigma \in T$ or $\neg\sigma \in T$ (or both).

Defn. T is consistent iff $\tau, \neg\tau \in T$ for no sentence τ.

Observation. For a given elementary language $\mathcal{L}$, there is exactly one complete, **in**consistent theory, viz., $Sent(\mathcal{L})$.

The Enumerability Theorem

Defn. An elementary language $\mathcal{L}$ is said (by Enderton) to be "reasonable" iff (i) $\mathcal{L}$ itself is effectively enumerable, and (ii) both $Pred(\mathcal{L}, n)$ and $Fn(\mathcal{L}, n)$ are decidable for each $n \in \omega$.

Enumerability Theorem. Let Γ be a decidable set of wffs in a "reasonable" language $\mathcal{L}$. Then the set $\{\varphi \in WFF(\mathcal{L}) \ : \ \Gamma \models \varphi\}$ is effectively enumerable.

Proof. The proof depends on the completeness theorem for the Hilbert system, however that might be demonstrated.
Since $\mathcal{L}$ is "reasonable," the set of all finite sequences of wffs of $\mathcal{L}$ is effectively enumerable. Since, given the decidability of Γ and of the logical axioms Λ, there is a decision procedure to determine whether a given finite sequence of wffs is a Hilbert proof. Delete from the effective enumeration of all finite sequences of wffs those that are not Hilbert proofs and write down the last wff of each proof. This yields an effective enumeration of the set $\{\varphi \in WFF(\mathcal{L}) \ : \ \Gamma \models \varphi\}$. ■

Some Easy Consequences

Proposition. A finitely axiomatizable theory is axiomatizable.

Corollary. An axiomatizable theory in a "reasonable" language is effectively enumerable.

Lemma. If Cn Σ is f.a., then there is a finite $\Sigma_0 \subseteq \Sigma$ s.t. Cn $\Sigma_0 \subseteq$ Cn Σ.

Proof. Assume that Cn Σ is f.a. Then, for some sentence τ, Cn $\tau =$ Cn Σ. Since $\Sigma \models \tau$, by compactness there is a finite $\Sigma_0 \subseteq \Sigma$ s.t. $\Sigma_0 \models \tau$. Hence,

$$\text{Cn } \tau \subseteq \text{Cn } \Sigma_0 \subseteq \text{Cn } \Sigma = \text{Cn } \tau,$$

whence Cn $\Sigma_0 =$ Cn Σ. ■

More Easy Consequences

Proposition. A decidable theory is effectively enumerable.

Lemma. A complete axiomatizable theory in a "reasonable" language is decidable.

Proof. There are two cases to consider: (i) T is consistent, and (ii) T is inconsistent.

Take case (ii) first. Then T is the set of all sentences of the language, which is decidable. So consider case (i). Since T is axiomatizable, it is effectively enumerable. To decide whether a sentence $\sigma \in T$, take the output of an effective enumeration of T: $\alpha_0, \alpha_1, \ldots, \alpha_n, \ldots$. In increasing order, ask, for each $n \in \omega$, whether $\alpha_n = \sigma$ or $\alpha_n = \neg\sigma$. Since T is consistent and complete, exactly one of the two must show up on the list. If it is σ, then $\sigma \in T$. If it is $\neg\sigma$, then $\sigma \notin T$. ■

A not so Easy Consequence

Craig's Theorem. An effectively enumerable theory is axiomatizable.

Proof sketch. Let $\alpha_0, \alpha_1, \ldots, \alpha_n, \ldots$ be an effective enumeration of T. If any outputted sentence has the form of some sentence σ conjoined with itself two or more times, replace it with simply σ. Let $\beta_0, \beta_1, \ldots, \beta_n, \ldots$ be the resulting effective enumeration. Now, for each n, let γ_n be β_n conjoined with itself $n+1$ times. We claim that $\Sigma =_{df} \{\gamma_n \mid n \in \omega\}$ axiomatizes T. Two things must be established: (i) Σ is decidable, and (ii) $\mathrm{Cn}\ \Sigma = T$.

For (i), to tell if $\tau \in T$, first determine whether it has the form of some sentence σ conjoined with itself $n+1$ times and then whether $\tau = \gamma_n$. For (ii), note that for each $n \in \omega$, γ_n is logically equivalent to α_n. ■

Theories with only Infinite Models

WE DID THIS ALREADY. SKIP!

Let $\mathcal{L}$ have a single 2-place relational predicate R. It's not hard to fashion theories having only infinite models without using identity. Here's a finitely axiomatizable case. Let σ be the conjunction of the following sentences.

$$\forall v_0 \neg R v_0 v_0$$
$$\forall v_0 \forall v_1 \forall v_2 (R v_0 v_1 \to R v_1 v_2 \to R v_0 v_2)$$
$$\forall v_0 \exists v_1 R v_0 v_1$$

Then Th σ has only infinite models.

INSTEAD CONSIDER ...

Non-standard Models of the Same Infinite Cardinality

Let $\mathfrak{N}$ be the standard model of Peano arithmetic, i.e., $\mathfrak{N} = (\mathbb{N}, 0, S, +, \cdot)$. We defer until later the question whether Th $\mathfrak{N}$ is the theory of Peano arithmetic. We know, however, by LST that Th $\mathfrak{N}$, although complete, has models of all infinite cardinalities, and thus models not isomorphic to $\mathfrak{N}$. But are there *countable* models of Th $\mathfrak{N}$ not isomorphic to $\mathfrak{N}$?

Define the new relational predicate $<$ by

$$n < m \text{ iff } \exists k : n + Sk = m.$$

Add a new constant c to the language of arithmetic, and add to Th $\mathfrak{N}$ the sentences: $0 < c, S0 < c, \ldots, S^n 0 < c, \ldots$. The resulting set is finitely satisfiable and hence, by compactness, satisfiable. By Löwenhiem-Skolem, it has a countable model. But any model of the resulting set contains an element larger than any natural number, and thus is not isomorphic to $\mathfrak{N}$.

Skolem's Paradox

The Löwenhiem-Skolem theorem entails that ZF (or ZFC) has a countable model.

Yet, there are sentences of the theory asserting the existence of uncountable sets (e.g., $\mathscr{P}(\mathbb{N})$).

How can that be?

Discussion.

The Łoś-Vaught Test

The preceding example motivates the following definition.

Defn. Let κ be a cardinal. T is κ-categorical iff any two models of T of cardinality κ are isomorphic.

Łoś-Vaught Test. Let T be a satisfiable theory in a countable language having no finite models. Then, if T is κ-categorical in some infinite cardinal κ, then T is complete.

Proof. Suppose T is κ-categorical and let $\mathfrak{A}$ and $\mathfrak{B}$ be any models of T. Since both are infinite, there are structures $\mathfrak{A}'$ and $\mathfrak{B}'$ of cardinality κ elementarily equivalent to $\mathfrak{A}$ and $\mathfrak{B}$, respectively. Since T is κ-categorical, $\mathfrak{A}'$ is isomorphic to $\mathfrak{B}'$. Thus, we have

$$\mathfrak{A} \equiv \mathfrak{A}' \simeq \mathfrak{B}' \equiv \mathfrak{B}.$$

So, any two models of T are elementarily equivalent, and so T is complete. ■

Application I. Algebraically Closed Fields of Characteristic 0

Exercise. Let $\mathcal{L} = \{0, 1, +, \cdot\}$ and let $\mathcal{F}$ be the class of structures for $\mathcal{L}$ that are algebraic fields. Show that Th $\mathcal{F}$ is finitely axiomatizable.

Exercise. Let $\mathcal{F}_0$ be the class of structures for $\mathcal{L}$ that are fields of characteristic 0. Show that Th $\mathcal{F}_0$ is axiomatizable.

Exercise. Let $\mathcal{F}_0^C$ be the class of structures for $\mathcal{L}$ that are algebraically closed fields of characteristic 0. Show that Th $\mathcal{F}_0^C$ is axiomatizable.

Lemma ex machina. Th $\mathcal{F}_0^C$ is categorical in any uncountable cardinality.

Application I (cont.)

Theorem. (i) Th $\mathcal{F}_0^C$ is complete. Also, (ii) the theory of the complex field is decidable.

Proof. (i) The language $\mathcal{L}$ of field theory is countable. Th $\mathcal{F}_0$, and, thus, Th $\mathcal{F}_0^C$ has no finite models. Furthermore, Th $\mathcal{F}_0^C$ is κ-categorical for any uncountable κ. Thus, by the Łoś-Vaught test, Th $\mathcal{F}_0^C$ is complete.
For part (ii), let $\mathfrak{C}$ be the complex field. Since $\mathfrak{C} \in \mathcal{F}_0^C$ and Th $\mathcal{F}_0^C$ is complete, it follows that Th $\mathfrak{C} =$ Th $\mathcal{F}_0^C$. And since Th $\mathcal{F}_0^C$ is complete and axiomatizable, it is decidable. Hence, Th $\mathfrak{C}$ is as well. ■

Theorem. Let $\mathfrak{R}$ be the real field. Th $\mathfrak{R}$ is decidable.

Proof. Tarksi. The Łoś-Vaught test does not apply since Th $\mathfrak{R}$ is not categorical in any infinite cardinality. ■

Application II. Theory of Dense Linear Orderings w/o Endpoints

Exercise. Let $\mathcal{L}$ contain a single 2-place predicate symbol $<$. Let $\mathcal{D}$ be the class of structures for $\mathcal{L}$ that are dense linear orderings without endpoints. Show that Th $\mathcal{D}$ is axiomatizable, in fact, finitely so.

Theorem (Cantor). Any two countable members of $\mathcal{D}$ are isomorphic.

Proof. Exercise (really!).

Theorem. Th $\mathcal{D}$ is complete and decidable.

Proof. $\mathcal{L}$ is countable and Th $\mathcal{D}$ has no finite models and is $\aleph_0$-categorical. By the Łoś-Vaught test, Th $\mathcal{D}$ is complete. Since it is axiomatizable, it is also decidable. ■

Scholium. $(\mathbb{R}, <_{\mathbb{R}})$ cannot be distinguished from $(\mathbb{Q}, <_{\mathbb{Q}})$ by any sentence or set of sentences involving only $<$.

Prenex Normal Form

Defn. We define by recursion when a wff is in prenex normal form.

1. If α is quantifier-free, then α is in prenex normal form.
2. If α is in prenex normal form, then so are $\forall x\alpha$ and $\exists x\alpha$ for any variable x.

So, a wff is in prenex normal form if it has the form

$$Q_1x_1 \cdots Q_nx_n\beta,$$

where
(i) each Q_i is a universal or an existential quantifier, and
(ii) β is quantifier free.

Prenex Normal Form Theorem. Every wff has a logical equivalent in prenex normal form.

Proof. Exercise

Finite Models and Decidability

Simple Fact. Every structure of cardinality n is isomorphic to a structure $\mathfrak{A}$ s.t. $|\mathfrak{A}| = n$.

Simple Fact. If the language is finite, then, for a given $n \in \omega$, there are only finitely many such structures with domain n. Put another way, there are only finitely many equivalence classes of structures of cardinality n, where the equivalence relation is isomorphism.

Simple Fact. If the language is finite, a finite structure $\mathfrak{A}$ can be fully described by a finite list. (diagram of $\mathfrak{A}$)

Fact. Let $\mathfrak{A}$ be a finite structure for a finite language, and let $s_\varphi : FV(\varphi) \to |\mathfrak{A}|$. Whether or not $\models_{\mathfrak{A}} \varphi\ [s_\varphi]$ is decidable.

Corollary. Th $\mathfrak{A}$ is decidable for a finite structure for a finite language.

Fact. The relation $\{(\sigma, n) \mid \sigma \text{ has a model of size } n\}$ is decidable.

Finite Models and Decidability (cont.)

Theorem. For a finite language, the set of sentences having a finite model is effectively enumerable.

Proof. Recall that

$$R =_{df} \{(\sigma, n) \mid \sigma \text{ has a model of size } n\}$$

is decidable. Let $\alpha_0, \alpha_1, \ldots$ be an effective enumeration of the sentences of the language. Informally, proceed as follows. Test whether α_0 has a 1-element model. Test whether α_1 has a 1-element model and α_0 has a 2-element model. And so on, listing α_n if it passes a test (unless already on the list). More formally:

see next slide

Code for the effective enumeration

```
n := 0;
while (n++ ≥ 0) {
    for i = 0 to n {
      for j = 1 to n + 1 {
        if (α_i, j) ∈ R,
        print α_i;
      }
    }
}
```

This code will produce a high degree of redundancy in the effective enumeration, which is OK. If you want, you can eliminate it with more complex code. ■

A Last Result re Finite Models

Lemma. Let Θ be the set of all sentences true in all finite structures of a finite language. Then, for any sentence σ, $\sigma \in \Theta^c$ iff $\neg\sigma$ has a finite model.

Proof. We prove the contrapositive in both directions. $(\Rightarrow)$ Suppose that $\neg\sigma$ has no finite models. Then σ is true in every finite model and hence $\sigma \in \Theta$. Thus, $\sigma \notin \Theta^c$. $(\Leftarrow)$ Suppose that $\sigma \notin \Theta^c$. Hence, $\sigma \in \Theta$ and thus σ is true in every finite structure. Thus, $\neg\sigma$ has not finite models. ■

Theorem. The relative complement Θ^c of Θ is effectively enumerable.

By the lemma, $\sigma \in \Theta^c$ iff $\neg\sigma$ has a finite model. We have a semi-decision procedure for the right-hand side, and thus for the left. And, a set is effectively enumerable iff there is a semi-decision procedure for membership. ■