A SYSTEM OF NATURAL DEDUCTION FOR ELEMENTARY LOGIC

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The system of natural deduction for elementary logic given here is chosen in such a way that a system of natural deduction for intuitionistic predicate logic (without identity)[1] is contained as a proper subsystem obtained simply by eliminating the classical rule for *reductio ad absurdum* (as well as the primitive rules for identity).

Whenever one wants to entertain this restriction, however, it is absolutely essential to keep in mind that it is not possible to move to a more parsimonious basis of logical constants, as is customarily done in elementary logic. Intuititionistic conjunction and disjunction cannot be reduced to negation and implication, as is the case with their classical counterparts, and similarly, existential (universal) quantification cannot be defined in terms of universal (existential) and negation. Indeed, it is more natural not to introduce negation as a primitive connective, but instead to work with *absudity* $\bot$ (a.k.a., 'bottom','contradiction', 'the false') as primitive and to define $\neg\varphi$ as $(\varphi \to \bot)$. This captures the difference between the intuitionistic and the classical understandings of negation in terms of the respectively different understandings of $\to$. Whatever one's choice of basis for classical logic, $\bot$ can be suitably defined by understanding it as an abbreviation for whatever happens to be one's favorite unsatisfiable formula.

However, we take $(\varphi \leftrightarrow \psi)$ to be a metalinguistic abbreviation for the object language formula

$$(\varphi \to \psi) \land (\psi \to \varphi)$$

in both intuitionistic and classical logic.

Before setting out the rules of inference, it is wise to establish some notation for uniform substitution and nomenclature for permissive substitution within the context of the rules.

1. Uniform Substitution and OK substitution.

If x is a variable, t a term, and φ a well-formed formula (of whatever elementary language is under consideration), then $[\varphi]^x_t$ is the result of uniformly substituting t for x wherever x occurs free in φ. More precisely, this is defined by recursion as follows.

Basis Cases: (1) If P is an n-ary predicate and $t_1, \ldots, t_n$ terms of the language, then

$$[Pt_1 \cdots t_n]^x_t = P[t_1]^x_t \cdots [t_n]^x_t,$$

where $[t_i]^x_t$ is the result of replacing x by t wherever x occurs in t_i. (This latter notion can be defined precisely by recursion, as well. *Exercise*.)

Date: February 2009.

[1]The status of identity and, more obviously, the status of non-identity, is a bit tricky in intuitionism. Henceforth, in dealing with intuitionistic predicate calculus we will assume that no identity predicate is available as a logical constant from the outset.

(2) If I is the identity symbol, then, for any terms t_1 and t_2,

$$[It_1t_2]^x_t = I[t_1]^x_t[t_2]^x_t.$$

Inductive Cases: Let φ and ψ be arbitrary well-formed formulas. Then, for the application of connectives,

$$\begin{aligned}
[\neg\varphi]^x_t &= \neg[\varphi]^x_t \\
[(\varphi \to \psi)]^x_t &= ([\varphi]^x_t \to [\psi]^x_t) \\
[(\varphi \wedge \psi)]^x_t &= ([\varphi]^x_t \wedge [\psi]^x_t) \\
[(\varphi \vee \psi)]^x_t &= ([\varphi]^x_t \vee [\psi]^x_t)
\end{aligned}$$

while, for universal quantification over any variable y,

$$[\forall y\varphi]^x_t = \begin{cases} \forall x\varphi & \text{if } x = y \\ \forall y[\varphi]^x_t & \text{otherwise,} \end{cases}$$

and similarly for existential quantification.

The above definition does not prohibit a variable in the substituted term t becoming captured by a quantifier expression upon substitution. Let φ, for example, be the formula $\exists yRyx$ with free variable x and t the term fy. Then $[\varphi]^x_t$ becomes the *sentence* $\exists yRyfy$. We want to exclude such behavior in inference rules involving uniform substitution (else we cannot vouchsafe their soundness). If such variable capture does not result upon substitution, we will say that the substitution is *OK*. More formally, for any variable x and term t,

(1) if φ is atomic, then $[\varphi]^x_t$ is OK;
(2) if $[\varphi]^x_t$ is OK, then so is $[\neg\varphi]^x_t$;
(3) if both $[\varphi]^x_t$ and $[\psi]^x_t$ are OK, then so are $[(\varphi \to \psi)]^x_t$, $[(\varphi \wedge \psi)]^x_t$, $[(\varphi \vee \psi)]^x_t$; and
(4) if $[\varphi]^x_t$ is OK, then $[\forall y\varphi]^x_t$ and $[\exists y\varphi]^x_t$ are OK provided y does not occur in t.

We now have what we need in order to formulate the rules of inference.

2. Inference Rules for Intuitionistic Predicate Logic

I assume some familiarity with the Fitch-style layout for indicating assumptions in subderivations and their discharge by reducing the level of nesting.[2] Hence, without further ado, we the rules for intuitionistic predicate logic may be stated as follows.

Modus Ponens ($\to$ Out)

| $(\varphi \to \psi)$
| .
| .
| φ
| .
| .
$\rhd$ | ψ

Conditional Derivation ($\to$In)

| | φ
| | ———
| | .
| | .
| | ψ
$\rhd$ | $(\varphi \to \psi)$

Right Simplification ($\wedge\text{Out}_R$)

| $(\varphi \wedge \psi)$
| .
| .
$\rhd$ | φ

[2] See either Barwise and Etchemendy or else Bergmann, Moore and Nelson for very accessible presentations.

Left Simplification ($\wedge$Out$_L$)

$$\begin{array}{l|l} & (\varphi \wedge \psi) \\ & \cdot \\ & \cdot \\ \triangleright & \psi \end{array}$$

Adjunction ($\wedge$In)

$$\begin{array}{l|l} & \varphi \\ & \cdot \\ & \cdot \\ & \psi \\ & \cdot \\ & \cdot \\ \triangleright & (\varphi \wedge \psi) \end{array}$$

Absurdity to Anything ($\bot$Out)

$$\begin{array}{l|l} & \bot \\ & \cdot \\ & \cdot \\ \triangleright & \varphi \end{array}$$

Repetition (Reiteration)

$$\begin{array}{l|l} & \varphi \\ & \cdot \\ & \cdot \\ \triangleright & \varphi \end{array}$$

Right Addition ($\vee$In$_R$)

$$\begin{array}{l|l} & \varphi \\ & \cdot \\ & \cdot \\ \triangleright & (\varphi \vee \psi) \end{array}$$

Left Addition ($\vee$In$_L$)

$$\begin{array}{l|l} & \varphi \\ & \cdot \\ & \cdot \\ \triangleright & (\psi \vee \varphi) \end{array}$$

Separation of Cases ($\vee$Out)

$$\begin{array}{l|l} & (\varphi \vee \psi) \\ & \cdot \\ & \cdot \\ & \begin{array}{|l} \varphi \\ \hline \cdot \\ \cdot \\ \theta \end{array} \\ & \cdot \\ & \cdot \\ & \begin{array}{|l} \psi \\ \hline \cdot \\ \cdot \\ \theta \end{array} \\ & \cdot \\ & \cdot \\ \triangleright & \theta \end{array}$$

Universal Instantiation ($\forall$Out)

$$\begin{array}{l|l} & \forall x\varphi \\ & \cdot \\ & \cdot \\ \triangleright & [\varphi]^x_t \end{array}$$

provided $[\varphi]^x_t$ is OK

Existential Generalization ($\exists$In)

$$\begin{array}{l|l} & [\varphi]^x_t \\ & \cdot \\ & \cdot \\ \triangleright & \exists x\varphi \end{array}$$

provided $[\varphi]^x_t$ is OK

Universal Derivation (∀In)

$$
\begin{array}{l|l}
 & \quad \begin{array}{|l}
\text{[declaration of } v\text{]} \\ \hline
\cdot \\
\cdot \\
[\varphi]^x_v
\end{array} \\
\rhd & \forall x \varphi
\end{array}
$$

Existential Instantiation (∃Out)

$$
\begin{array}{l|l}
 & \exists x \varphi \\
 & \cdot \\
 & \cdot \\
 & \cdot \\
 & \quad \begin{array}{|l}
\text{[declaration of } n\text{]} \\
[\varphi]^x_n \\ \hline
\cdot \\
\cdot \\
\cdot \\
\psi
\end{array} \\
\rhd & \psi
\end{array}
$$

provided n does not occur in ψ.

3. Classical Rules of Inference for Elementary Logic

All intuitionistic rules are classical rules, as well as the following.

Classical Reductio (¬Out)

$$
\begin{array}{l|l}
 & \quad \begin{array}{|l}
\neg\varphi \\ \hline
\cdot \\
\cdot \\
\cdot \\
\bot
\end{array} \\
\rhd & \varphi
\end{array}
$$

When identity is given as a logic constant, the rules governing it are:

Self Identity (= In)

$$
\begin{array}{l|l}
 & \cdot \\
 & \cdot \\
\rhd & t = t
\end{array}
$$

and

Leibniz's Law (= Out)

$$
\begin{array}{l|l}
 & t_1 = t_2 \\
 & \cdot \\
 & \cdot \\
 & \varphi \\
 & \cdot \\
 & \cdot \\
\rhd & \varphi[t_1/t_2]
\end{array}
$$

where $\varphi[t_1/t_2]$ is the result of replacing the term t_1 for the term t_2 in φ zero or more times, subject to the condition that the replacement is *OK* in the sense analogous to that for uniform substitution.

4. Derived Rules of Inference

I want first to show that there is a derived intuitionistic form of reductio. Recall that $\neg\varphi$ is a metalinguistic abbreviation for $(\varphi \to \bot)$. As an instance of Conditional Derivation, we have

$$
\begin{array}{l|l}
 & \quad \begin{array}{|l} \varphi \\ \hline \cdot \\ \cdot \\ \bot \end{array} \\
\triangleright & (\varphi \to \bot)
\end{array}
$$

which is clearly just the form of reductio:

Intuitionistic Reductio ($\neg$In)

$$
\begin{array}{l|l}
 & \quad \begin{array}{|l} \varphi \\ \hline \cdot \\ \cdot \\ \cdot \\ \bot \end{array} \\
\triangleright & \neg\varphi
\end{array}
$$

Secondly, one instance of Modus Ponens, is the form

$$
\begin{array}{l|l}
 & (\varphi \to \bot) \\
 & \cdot \\
 & \cdot \\
 & \varphi \\
 & \cdot \\
 & \cdot \\
\triangleright & \bot
\end{array}
$$

which yield immediately a derived rule for ($\bot$-In), viz.,

Contradiction to Absurdity ($\bot$In)

$$
\begin{array}{l|l}
 & \varphi \\
 & \cdot \\
 & \cdot \\
 & \neg\varphi \\
 & \cdot \\
 & \cdot \\
\triangleright & \bot
\end{array}
$$

Now, there is a host of other familiar rules of inference that can be obtained as derived rules given the primitive rules stated above.

Your task to derive whatever rules additional rules you a fond of using from the rules given so far. As an additional task, you should try to sort these rules into those that are intuitionistically acceptable and those that are not. Leaving aside for the moment the status of the rules governing identity, you can establish that a derived rule is intuistionistically acceptable if your derivation of it does not appeal to Classical Reductio (or to any rule previously derived by an appeal to Classical Reductio). To show that a derived is not intuitionaistically acceptable, it suffices to show that adoption of the rule, together with intuitionisitic rules, leads to a derivation of Classical Reductio. You should do your best to sort out from the familiar moves you have been taught those that are intuistionistically warranted from those that presuppose Classical Reduction.

Happy hunting!

Shortly, I will post a list of the most commonly appearing rules, with a sorting into the two categories.