SYNOPSIS OF PETER SMITH'S *INTRO TO GÖDEL'S THEOREMS*

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1. Some "Easy" Incompleteness Theorems

Defn An interpreted language is *sufficiently expressive* iff (i) every decidable two-place relation on numbers is expressible / definable (in the standard model), and (ii) it can form wff's that quantify over numbers.

Lemma If W is an effectively enumerable set (of numbers), then there is a decidable (numerical) relation R such that $n \in W$ just in case $\exists x$ s.t. $\langle x, n\rangle \in R$.
Proof: R is just the enumerating function $f(0), f(1), \ldots$ as a a set of ordered pairs.

Lemma W is an effectively enumerable set (of numbers) iff W is the domain of some (numerical) algorithm Π.
Proof. (only if) Let $f(0), f(1), \ldots$ be an effective enumeration of W. Construct Π as follows. For any given $n \in W$, crank out the enumeration until for some m, $f(m) = n$. Assign 0 to n.
(if) Given algorithm Π, spend one step on 0, 2 steps on 0 and 1, respectively, etc., in each case writing down the number if the algorithm completes.

Theorem There is an effectively enumerable set (of numbers) whose complement (in the natural numbers) is not effectively enumerable.
Proof. Enumerate the set of effective procedures (numerical algorithms). By the last theorem, this is equivalent to an effective enumeration $W_0, W_1, \ldots$ of all recursively enumerable sets. Let $K = \{n \mid n \in W_n\}$. K is effectively enumerable by computing down the anti-diagonals, putting $f_e(n) = e$ on the list whenever it is encountered. (Note: this is not the same as Smith's proof, which I'm not sure I get.) $\bar{K}$ is not effectively enumerable. Otherwise $\bar{K} = W_n$ for some n. Then $n \in W_n$ iff $n \notin \bar{K}$, i.e. $n \in \bar{K}$ iff $n \notin \bar{K}$.

The Semantic Theorem The set of truths in the intended interpretation of a sufficiently expressive arithmetic language is not effectively enumerable.
Proof. (i) Take K from the last theorem. By the first lemma, there is an effective relation R s.t. for some x, $\langle x, n\rangle \in R$ iff $n \in K$. (ii) Since the language is sufficiently expressive, there is a formula ρ such that $\langle m, n\rangle \in R$ iff $\rho(\bar{m}, \bar{n})$ is true. (iii) Let N be a formula that defines the set of natural numbers in the intended model. Then $n \in K$ iff $\exists x(N(x) \wedge \rho(x, \bar{n}))$ is true. (iv) Hence $\neg\exists x(N(x) \wedge \rho(x, \bar{n}))$ defines the set $\bar{K}$. (v) Now, for the sake of argument, suppose that the truths of the intended model are effectively enumerable. This yields an

effective procedure for enumerating $\bar{K}$, viz., go through the effective enumeration of truths, writing down k whenever $\neg\exists x(N(x) \wedge \rho(x, \bar{k}))$ is encountered. But it was already shown that $\bar{K}$ is not effectively enumerable.

Corollary The set of sentences true in the intended model is not axiomatizable.

Corollary Any true axiomatizable arithmetic theory is incomplete.

Definition An arithmetic theory T is *sufficiently strong* iff T represents (captures) all effectively decidable numerical properties.

The Syntactic Theorem Any consistent, sufficiently strong arithmetic theory is undecidable. Proof. Effectively enumerate the set of wff's of the language with exactly x free: $\varphi_0(x), \varphi_1(x), \ldots$. Define the set D as follows. $n \in D$ iff $T \vdash \neg\varphi_n(\bar{n})$, where T is the theory in question. Suppose T is decidable. Then D is decidable, and hence representable by some wff δ with exactly x free. So, for some n, $\delta = \varphi_n$. Now $n \in D$ iff $T \vdash \neg\varphi_n(\bar{n})$ iff $T \vdash \neg\delta(\bar{n})$ iff $n \notin D$.

Corollary Any consistent, sufficiently strong, axiomatizable arithmetic theory is incomplete. Pf. Effectively enumerate the theory T. If complete, running through the effective enumeration gives us a decision procedure for membership in T.

2. Some Arithmetic Theories

2.1. Baby Arithmetic BA, or perhaps Calculator Arithmetic.

- no variables or quantifiers
- Axiom Schemata
  1. $0 \neq S\xi$
  2. $S\xi = S\eta \rightarrow \xi = \eta$
  3. $\xi + 0 = \xi$
  4. $\xi + S\eta = S(\xi + \eta)$
  5. $\xi \cdot 0 = 0$
  6. $\xi \cdot S\eta = \xi \cdot \eta + \xi$
- Lemmata
  1. If $\sigma = \tau$ is true, then $BA \vdash \sigma = \tau$.
  2. If $\sigma = \tau$ is false, then $BA \vdash \sigma \neq \tau$.
- BA is complete

2.2. Robinson's Q.

2.2.1. *Axioms.*

1. $\forall x\, 0 \neq Sx$
2. $\forall x \forall y\, (Sx = Sy \rightarrow x = y)$
3. $\forall x\, (x \neq 0 \rightarrow \exists y\, x = Sy)$
4. $\forall x\, x + 0 = x$

(5) $\forall x \forall y\ x + Sy = S(x + y)$
(6) $\forall x\ x \cdot 0 = 0$
(7) $\forall x \forall y\ x + Sy = x \cdot y + x$

2.2.2. *Incompleteness.* We show that $Q \nvdash \forall x\ 0 + x = x$ by constructing a model that satisfies Q but not this last sentence. Let $N^* = N \cup \{a, b\}$, where a and b behave as follows in the extensions of the operations.

$$S^*(a) = a,\ S^*(b) = b$$

$$a +^* n = a,\ b +^* n = b,\ x +^* a = b,\ x +^* b = a$$

$$a \cdot^* 0 = 0,\ b \cdot^* 0 = 0$$

$$a \cdot^* x = b,\ b \cdot^* x = a \text{ if } x \neq 0$$

$$n \cdot^* a = b,\ n \cdot^* b = a$$

Also, given the intended model, $Q \nvdash \neg \forall x\ 0 + x = x$. Hence, Q is incomplete.

2.2.3. *Total ordering.* The formula $\exists z\ z + x = y$ defines the total ordering $\leq$ in the standard model. This formula also represents $\leq$ in Q. (Note: the formula $\exists z\ x + z = y$ also defines $\leq$ in the standard model, but a conjecture is that it fails to represent it in Q.)
Proof. Need to show

(1) $m \leq n$ iff $Q \vdash \exists z\ z + \bar{m} = \bar{n}$
(2) $m \nleq n$ iff $Q \vdash \neg \exists z\ z + \bar{m} = \bar{n}$

For (1), suppose that $m \leq n$ and specifically that $p + m = n$. Use the result that BA is complete and that Q extends BA to get that $Q \vdash \bar{p} + \bar{m} = \bar{n}$. Now use existential generalization. For (2), take case $2 \nleq 1$ for illustration. Suppose that $\exists z\ z + SS0 = S0$ and derive a contradiction in Q. Try to render the argument general by induction.

Q is also *order adequate* in that the following derivability relations hold.

(1) $Q \vdash \forall x\ 0 \leq x$
(2) For each n, $Q \vdash \forall x\,(x = 0 \vee x = 1 \vee \ldots \vee x = \bar{n} \rightarrow x \leq \bar{n})$
(3) For each n, $Q \vdash \forall x\,(x \leq \bar{n} \rightarrow x = 0 \vee x = 1 \vee \ldots \vee x = \bar{n})$
(4) For each n, if $Q \vdash \varphi(0)$, $Q \vdash \varphi(1)$, …, and $Q \vdash \varphi(\bar{n})$, then $Q \vdash (\forall x \leq \bar{n})\varphi(x)$
(5) For each n, if $Q \vdash \varphi(0)$, or $Q \vdash \varphi(1)$, or …, or $Q \vdash \varphi(\bar{n})$, then $Q \vdash (\exists x \leq \bar{n})\varphi(x)$.
(6) For each n, $Q \vdash \forall x(\bar{n} \leq x \rightarrow \bar{n} \leq Sx)$
(7) For each n, $Q \vdash \forall x(\bar{n} \leq x \rightarrow \bar{n} = x \vee S\bar{n} \leq x)$
(8) For each n, $Q \vdash \forall x(x \leq \bar{n} \vee \bar{n} \leq x)$ (Conjecture $Q \nvdash \forall x \forall y(x \leq y \vee y \leq x.)$)
(9) For each $n > 0$, if $Q \vdash (\forall x \leq \overline{n-1})\varphi(x)$, then $Q \vdash (\forall x \leq \bar{n})(x \neq \bar{n} \rightarrow \varphi(x))$

The proofs are exercises.

2.2.4. Δ_0, Σ_1, *and* Π_1 *Formulae.*

Defn A wff is Δ_0 atomic iff it has the form $\tau = \rho$ or $\tau \leq \rho$ for terms τ and ρ.

Defn The set of Δ_0 formulas is given as follows.

1. Any Δ_0 atomic wff is Δ_0.
2. If φ and ψ are Δ_0, then so are $\neg\varphi$, $(\varphi \to \psi)$, etc.
3. If φ is Δ_0, then so are $(\forall x \leq \kappa)\varphi$ and $(\exists x \leq \kappa)\varphi$, where x is any variable free is φ and κ is a numeral or else a variable distinct from x.

Defn A wff is strictly Σ_1 iff it has the form $\exists x_0 \ldots \exists x_n \varphi$ where φ is Δ_0 and $x_0, \ldots, x_n$ are free in φ.

Defn A wff is Σ_1 iff it is logically equivalent to a strictly Σ_1 wff.

Defn A wff is strictly Π_1 iff it has the form $\forall x_0 \ldots \forall x_n \varphi$ where φ is Δ_0 and $x_0, \ldots, x_n$ are free in φ.

Defn A wff is Π_1 iff it is logically equivalent to a strictly Π_1 wff.

Observations

1. The negation of a Δ_0 wff is Δ_0. (trivial)
2. The negation of a Σ_1 is Π_1 and vice-versa. (trivial)
3. Every Δ_0 wff is both Σ_1 and Π_1. (Let φ be Δ_0. Take $\varphi \wedge x = x$. Both the universal and existential generalizations of this are equivalent to φ.)
4. The set of true Δ_0 sentences is decidable. (Use induction on the definition of Δ_0.)

Defn Theory T is Γ-sound iff, for any Γ-sentence σ, if $T \vdash \sigma$, then σ is true.

Defn Theory T is Γ-complete iff, for any Γ-sentence σ, if σ is true, then $T \vdash \sigma$.

Lemma. Q correctly decides every Δ_0 atomic sentence. I.e., if δ is true, then $Q \vdash \delta$, and if δ is false, then $Q \vdash \neg\delta$.
Proof. For the case of equations, we are done (since $BA \subseteq Q$). Hence, suppose we have an inequality $\tau_1 \leq \tau_2$. Let n_1 and n_2 be the denotata of these, respectively. The equations $\tau_1 = \overline{n_1}$ and $\tau_2 = \overline{n_2}$ are derivable in Q. Since $\leq$ is representable in Q, $\overline{n_1} \leq \overline{n_2}$ is correctly decided.

Lemma. Q correctly decides every Δ_0 sentence.
Proof. By induction on the defn of Δ_0. The preceding lemma handles the basis case. The case of wffs formed by connectives is trivial. That leaves us with bounded quantification. Suppose $(\forall x \leq n)\varphi$ is true. Then, by this and the inductive hypothesis $Q \vdash \varphi(\bar{m})$ for all $m \leq n$. Since Q is order adequate, it follows that $Q \vdash (\forall x \leq n)\varphi$. So, suppose $(\forall x \leq n)\varphi$

is false. Then $\varphi(\bar{m})$ is false for some $m \leq n$, and hence, by the inductive hypothesis, $Q \vdash \neg\varphi(\bar{m})$. From this it follows that $Q \vdash \neg(\forall x \leq n)\varphi$. The case of bounded existential quantification is similar.

Theorem. Q is Σ_1 complete.
Proof. We need to show only that if $\exists x_0 \ldots \exists x_n \varphi$ is true, then $Q \vdash \exists x_0 \ldots \exists x_n \varphi$. So, suppose this sentence true. Then there exist $m_1, \ldots, m_n$ such that $\varphi(\overline{m_1}, \ldots, \overline{m_n})$ is a true Δ_0 sentence. Hence $Q \vdash \varphi(\overline{m_1}, \ldots, \overline{m_n})$, and by existential generalization $Q \vdash \exists x_0 \ldots \exists x_n \varphi$. This shows that Q proves all true *strictly* Σ_1 sentences. But, if σ is Σ_1, it is equivalent to a strictly Σ_1 sentence of the above form.

N.B. We need to be very careful with the next two results. Smith states the first of them as follows.

Theorem. A Π_1 sentence φ is true iff $Q \nvdash \neg\varphi$, i.e., iff φ is consistent with Q.
Proof. Equivalently, φ is false iff $\neg\varphi$ is true iff $Q \vdash \neg\varphi$, i.e., iff φ is inconsistent with Q.

But what if Q is inconsistent? Then $Q \vdash \neg\varphi$ since Q proves everything. Which, by the theorem, entails that the Π_1 sentence ϕ is false, no matter what σ says. We have only the following.

Theorem. For any Π_1 sentence φ, if $Q \nvdash \neg\varphi$, then φ is true.
Proof. Contrapositively: Suppose φ false. Then $\neg\varphi$ is a true Σ_1 sentence. Since Q is Σ_1 complete, $Q \vdash \neg\varphi$.

Note that we cannot work backward and infer from $Q \vdash \neg\varphi$ that $\neg\varphi$ is true. For if Q is inconsistent, it proves *all* Σ_1 formulas, and not just those that are true.

Theorem. Let T be an extension of Q. Then T is consistent iff T is Π_1-sound.
Proof. Suppose that T is *inconsistent*. Then for any *false* Π_1 sentence φ, $T \vdash \varphi$, and hence T is Π_1 *unsound*. For the other direction, suppose that T is consistent. Suppose also that it is not Π_1 sound. Thus, for some Π_1 sentence φ, $T \vdash \varphi$ but φ is false. Hence $\neg\varphi$, which is Σ_1, is true. Now note that since Q is Σ_1 complete and T extends Q, T is also Σ_1 complete. Thus, $T \vdash \neg\varphi$, rendering T inconsistent.

Note that PA extends Q, so that PA is consistent iff it is Π_1-sound. Now, it seems an easy route to consistency to say that, since the Peano axioms are all true in the standard model, that PA is Π_1 sound, and ergo consistent. But note on the other hand, that without any of the subtleties of the theorem, we don't know that PA is sound without knowing that it is consistent.

3. Peano Arithmetic (PA)

3.1. **Some Boring Details.**

Notational Tidbit: Add to Q the induction scheme for only Δ_0 wffs. The resulting system is called $I\Delta_0$.

Sometimes Σ_1 wffs are defined as a *single* existential quantifier followed by a Δ_0 wff. This is because of the following.

Lemma. In any theory which extends Q and has induction for Δ_0 wffs, a Σ_1 wff starting with $n > 1$ unbounded existential quantifiers is provably equivalent to a Σ_1 wff starting with a single unbounded existential quantifier.
Proof. The trick is this. Let $n = 2$ and consider (i) $\exists x \exists y \varphi(x, y)$. Under the hypothesis we replace it with (ii) $\exists z(\exists x \leq z)(\exists y \leq z)\varphi(x, y)$. To derive (ii) from (i), first existentially instantiate to $\varphi(a, b)$. Then, since $I\Delta_0 \vdash \forall x \forall y(x \leq y \vee y \leq x)$, we have by universal instantiation $a \leq b \vee b \leq a$. Using separation of cases, assume $a \leq b$. Then we have $a \leq b \wedge b \leq b \wedge \varphi(a, b)$. Existentially generalize to $\exists x \exists y(x \leq b \wedge y \leq b \wedge \varphi(x, y))$. Now existentially generalize on the remaining occurrences of b to $\exists z \exists x \exists y(x \leq z \wedge y \leq z \wedge \varphi(x, y))$. Using notational abbreviation, this is just (ii). To derive (i) from (ii), take (ii) in the expanded form just given and distribute the existential quantifiers over the conjunctions, dropping the idle quantifier on the last conjunct.

Note: the key is the availability of the dichotomy proposition for $\leq$. So why not make the claim for Q + Dichotomy?

3.2. **Axiomatization of PA.**

Axiom 1: $\forall x(Sx \neq 0)$
Axiom 2: $\forall x \forall y(Sx = Sy \rightarrow x = y)$
Axiom 3: $\forall x(x + 0 = 0)$
Axiom 4: $\forall x \forall y(x + Sy = S(x + y))$
Axiom 5: $\forall x(x \cdot 0 = 0)$
Axiom 6: $\forall x \forall y(x \cdot Sy = x \cdot y + x)$
Induction Schema: For every wff $\varphi(x)$ with x or any other variables free, any universal generalization of

$$\{\varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(Sx))\} \rightarrow \forall x \varphi(x)$$

is an axiom.

4. Primitive Recursive (p.r.) Functions

4.1. **Definition.**

(1) The successor function S is p.r.
(2) The zero function $Z(n) = 0$ is p.r.
(3) For each n and $i \leq n$, the i-th projection function $I^n_i(x_1, \ldots, x_i, \ldots, x_n) = x_i$ is p.r.
(4) if $h(x_1, \ldots, x_i, \ldots x_n)$ and $g(\vec{y})$ are p.r., then $h(x_1, \ldots, g(\vec{y}), \ldots x_n)$ is p.r.
(5) if $g(\vec{x})$ and $h(\vec{x}, n, z)$ are p.r., then

- $f(\vec{x}, 0) = g(\vec{x})$
- $f(\vec{x}, S(n)) = h(\vec{x}, n, f(\vec{x}, n))$

is p.r.

4.2. **Primitive Recursive Functions, Programming and Computability.** Quite obviously any p.r. function is computable. Less obvious is that any p.r. function can be programmed using previously programmed p.r. functions and nested *for* loops.

We can argue that there are computable functions that are not p.r. as follows. Effectively enumerate the p.r. functions $f_0, f_1, \ldots$. Now define $F(n) = S(f_n(n))$. It's computable, yet $F \neq f_n$ for any n.

4.3. **Some Rudimentary Examples.**

4.3.1. *predecessor.*

$$\begin{aligned} P(0) &= 0 \\ P(Sn) &= n \end{aligned}$$

4.3.2. *truncated subtraction.*

$$\begin{aligned} x \dot{-} 0 &= x \\ x \dot{-} Sy &= P(x \dot{-} y) \end{aligned}$$

4.3.3. *absolute value.*

$$\mid x - y \mid = (x \dot{-} y) + (y \dot{-} x)$$

4.4. **Basic Tools for Further Development.**

Defn. A property or relation is p.r. iff it's characteristic function is p.r.

N.B. Smith reverses 1 and 0 in the definition of characteristic function.

Defn. $f(n) = (\mu x \leq n)P(x)$ is the function whose value at n is the least $x \leq n$ s.t. $P(x)$ if such an x exists or else returns n.

Defn. A function f is said to be defined by cases if for $k + 1$ other functions

$f(x) = f_1(x)$ if $P_1(x)$
$\vdots$
$f(x) = f_k(x)$ if $P_k(x)$
$f(x) = a$ otherwise.

If the P_i's and f_i's are p.r., then f is said to be defined by cases from p.r. functions.

Defn. sg is the p.r. function defined

$$\begin{aligned} sg(0) &= 0 \\ sg(Sn) &= SZ(sg(n)) \end{aligned}$$

and $\overline{sg}$ is the p.r. function defined

$$\begin{aligned}\overline{sg}(0) &= S(0)\\ \overline{sg}(Sn) &= Z(\overline{sg}(n)).\end{aligned}$$

Lemma A. If $f(x_1, \dots, x_n)$ is p.r., then the relation $\{\langle x_1, \dots, x_n, y\rangle \mid y = f(x_1, \dots, x_n)\}$ is p.r.
Proof. Let c be the characteristic function of f.

$$c(x_1, \dots, x_n, y) = \overline{sg}(\mid y - f(x_1, \dots, x_n) \mid)$$

and is thus p.r.

Lemma B. Any truth functional combination of p.r. properties and relations is p.r.
Proof: For $\neg$ use $\overline{sg}$. For $\vee$ use $sg(c_1 + c_2)$, where c_1 and c_2 are the characteristic values of the disjunct properties.

Lemma C. Any property or relation defined from a p.r. property or relation by bounded quantifications is also p.r.
Proof. This has the form

$$P(n) =_{df} (\forall x \leq n)Q(x)$$

or

$$P(n) =_{df} (\exists x \leq n)Q(x).$$

Let q be the characteristic function of $Q(x)$ and p that of $P(x)$. Then

$$p(n) = sg(\Pi_0^n q(x))$$

or

$$p(n) = sg(\Sigma_0^n q(x)).$$

(The cases $P(n) =_{df} (\forall x \leq f(n))Q(x)$ and $P(n) =_{df} (\exists x \leq f(n))Q(x)$. are obvious from this.) We still have to put these in a form that is manifestly p.r. For the first we have

$$\begin{aligned}p(0) &= q(0)\\ p(Sn) &= h(n, p(n)),\end{aligned}$$

where $h(n, y) = q(Sn) \cdot y$ is the composition of p.r. functions. For the second we have first the preliminary function

$$\begin{aligned}p'(0) &= q(0)\\ p'(Sn) &= h(n, p'(n)),\end{aligned}$$

where $h(x, y) = Sx + y$. Then p is the composition $p(n) = sg(p'(n))$.

Lemma D. If P is a p.r. property, then the function $(\mu x \leq n)P(x)$ is p.r.
Proof. Let $f(n)$ be the function. One way of describing it is as follows.

$$f(n) = \begin{cases} m & \text{if } Q(m) \text{ and } (\forall x < m)\neg Q(x)\\ n & \text{otherwise}\end{cases}$$

The "flip-flop" of the clause for m is captured by the function

$$g(m) = \overline{sg}\{q(m) \cdot \Pi_0^{m-1}\overline{sg}(q(x))\}.$$

This we know is p.r. Now let

$$f(n) = g(0) + g(1) + \cdots + g(n-1).$$

We know from the last lemma that this is also p.r., and f is the function sought.

Lemma E. Any function defined by cases from other p.r. functions is also p.r.
Proof.

$$f(n) = p_1(n) \cdot f_1(n) + \cdots + p_k(n) \cdot f_k(n) + \overline{sg}(p_1(n)) \cdot \cdots \cdot \overline{sg}(p_k(n)) \cdot a.$$

4.5. **Further Developments.** All the following are p.r.

- **R1:** $m = n$, $m < n$ $m \leq n$
- **R2:** $m \mid n$ (m divides n)
- **R3:** $Prime(n)$
- **R4:** $\pi(n)$ (the function which gives the n-th prime)
- **R5:** $exp(n, i)$, i.e., the exponent (which may be zero) of the i-th prime in the prime factorization of n
- **R6:** $len(n)$, i.e., the number of distinct prime factors of n (with $len(0) = len(1) = 0$).

Proofs of the various items.

R1. $m = n$ is p.r. since it is the identity function viewed as a relation (Lemma A). Specifically, it's characteristic function is given by $\overline{sg}(\mid m - n \mid)$.

$m < n$ is p.r. since it is given by the relation $(\exists v < n)(v + m = n)$. Alternatively, its characteristic function is $sg(n \dot{-} m)$.

$m \leq n$ is a boolean combination of the preceding two relations.

R2. $m \mid n$ is given by $\exists x \leq n(0 < x \wedge 0 < m \cdot x = n)$. (We need the < 0 clauses to rule out $0 \mid 0$)

R3. $Prime(n)$ is given by $n \neq 1 \wedge (\forall x \leq n)(1 < x \wedge x < n \rightarrow x \not| n)$.

R4.

$$\begin{aligned} \pi(0) &= 2 \\ \pi(Sn) &= (\mu x \leq n^n + 2)(\pi(n) < x \wedge Prime(x)) \end{aligned}$$

R5. $exp(n, i) = (\mu x \leq n)(\pi(i)^x \mid n \wedge \pi(i)^{x+1} \not| n)$

R6. Just to be clear, this is *not* the length of the prime expansion, including zero exponents, up through the last non-zero exponent. It is the number of non-zero exponents in the prime expansion of n. Towards this end, note that $(Prime(x) \wedge x \mid n)$ defines the primes that divide n. However, since later, no symbol will have the Gödel number 0, $len(n)$ does give us the length of the expression whose Gödel number is n. It is clearly p.r., so let p be its p.r. characteristic function. Now $len(n) = p(0, n) + p(1, n) + p(2, n) + \cdots + p(n, n)$.

In order to express this manifestly as a p.r. function, let

$$\begin{aligned} l(x,0) &= p(0,x) \\ l(x,Sy) &= p(Sy,x)+l(x,y). \end{aligned}$$

Then set $len(n) = l(n,n)$

5. CAPTURING P.R. FUNCTIONS

Capturing/representing a function f as a relation requires there exists a $\varphi(x,y)$ such that

- if $f(n)=m$, then $T \vdash \varphi(\bar{n},\bar{m})$, and
- if $f(n)\neq m$, then $T \vdash \neg\varphi(\bar{n},\bar{m})$.

You might want that T also captures that f is a function, i.e.,

(1) for each n, $T \vdash \exists! y\varphi(\bar{n},y)$
(2) if $f(n)=m$, then $T \vdash \varphi(\bar{n},\bar{m})$, and
(3) if $f(n)\neq m$, then $T \vdash \neg\varphi(\bar{n},\bar{m})$.

Observation If

(1) for each n, $Q \vdash \exists! y\varphi(\bar{n},y)$, and
(2) if $f(n)=m$, then $Q \vdash \varphi(\bar{n},\bar{m})$

then (3) if $f(n)\neq m$, then $Q \vdash \neg\varphi(\bar{n},\bar{m})$.
Proof. Suppose (1) and (2) and that $f(n)\neq m$. Obviously we have $Q \vdash \varphi(\bar{n},\overline{f(n)})$. (1) unpacks as $Q \vdash \exists y(\varphi(\bar{n},y) \wedge \forall x(\varphi(\bar{n},x) \rightarrow x=y))$. By existential instantiation, $Q \vdash (\varphi(\bar{n},a) \wedge \forall x(\varphi(\bar{n},x) \rightarrow x=a))$. By simplification, $Q \vdash \forall x(\varphi(\bar{n},x) \rightarrow x=a)$. By universal instantiation, $Q \vdash (\varphi(\bar{n},\overline{f(n)}) \rightarrow \overline{f(n)}=a)$. By modus ponens, $Q \vdash \overline{f(n)}=a$. By substitution of identicals, $Q \vdash \forall x(\varphi(\bar{n},x) \rightarrow x=\overline{f(n)})$. By universal instantiation, $Q \vdash (\varphi(\bar{n},\bar{m}) \rightarrow \bar{m}=\overline{f(n)})$. Now, since Q includes BA, which decides all quantifier free sentences, $Q \vdash \bar{m}\neq\overline{f(n)}$. By modus tollens, we have $Q \vdash \neg\varphi(\bar{n},\bar{m})$.

Hence, if T includes BA, sometimes only (1) and (2) are used in the defn of capturing/representing as a function.

Observation If T includes BA, the following condition is necessary and sufficient to show that $\varphi(x,y)$ captures $f(x)=y$ in T as a function.

if $f(n)=m$, then $T \vdash \forall x(\varphi(\bar{n},x) \leftrightarrow x=\bar{m})$.

(Simpler still would be $T \vdash \forall x(\varphi(\bar{n},x) \leftrightarrow x=\overline{f(n)})$.)

Proof. Sufficiency is an easy exercise. Techniques from the above proof can be used to show necessity.

Yet a stronger notion is this.
Defn. T fully captures (represents) the function f with φ iff

- $T \vdash \forall x \exists! y \varphi(x, y)$
- if $f(n) = m$, then $T \vdash \varphi(\bar{n}, \bar{m})$
- if $f(n) \neq m$, then $T \vdash \neg\varphi(\bar{n}, \bar{m})$.

If T is at least as strong as Q, it suffices to show that f is representable in T as a relation, in order to show that f is representable in T as a function and even that f is *fully* representable in T. This is because of the following two lemmata.

Lemma. Suppose T is at least as strong as Q. Then if $\varphi(x, y)$ represents f as a relation in T, then $\tilde{\varphi}(x, y) =_{df} \varphi(x, y) \wedge (\forall z \leq y)(\varphi(x, z) \rightarrow z = y)$ represents f as a function in T.

Proof. We need to show two things, viz.:

(1) for all n, $T \vdash \exists! y \tilde{\varphi}(\bar{n}, y)$, and
(2) if $f(n) = m$, then $T \vdash \tilde{\varphi}(\bar{n}, \bar{m})$.

We first show (2). Suppose that $f(m) = n$. Then (a) $T \vdash \varphi(\bar{n}, \bar{m})$. Furthermore, (b) for all $k < m$, $T \vdash \neg\varphi(\bar{n}, \bar{k})$. Hence, from (a) and (b), follows (c) for all $k \leq m$, $T \vdash \varphi(\bar{n}, \bar{k}) \rightarrow \bar{k} = \bar{m}$. Since Q is order adequate, (c) entails (d) $T \vdash (\forall z \leq m)(\varphi(\bar{n}, z) \rightarrow z = \bar{m})$. Putting this together with (a) yields $T \vdash \varphi(\bar{n}, \bar{m}) \wedge (\forall z \leq m)(\varphi(\bar{n}, z) \rightarrow z = \bar{m})$, which is just the statement that $T \vdash \tilde{\varphi}(\bar{n}, \bar{m})$.

This leaves (1) to be shown. Explicitly, we need to show that, for all n,

$$T \vdash \exists y(\tilde{\varphi}(\bar{n}, y) \wedge \forall z(\tilde{\varphi}(\bar{n}, z) \rightarrow z = y)).$$

Fix n. Since $T \vdash \tilde{\varphi}(\bar{n}, \bar{m})$, it suffices to show for arbitrary a that $T \vdash (\tilde{\varphi}(\bar{n}, a) \rightarrow a = \bar{m})$. Working within T, suppose $\tilde{\varphi}(\bar{n}, a)$, i.e., $\varphi(\bar{n}, a) \wedge (\forall z \leq a)(\varphi(\bar{n}, z) \rightarrow z = a)$. Now, since T is order adequate, $T \vdash a \leq \bar{m} \vee \bar{m} \leq a$. On the one hand, suppose that $a \leq \bar{m}$. By (d) above, $a = \bar{m}$, the desired consequence. So, on the other hand, suppose that $\bar{m} \leq a$. Then, by the second conjunct of our supposition, $\bar{m} = a$. So, in either case, we are done.

The second lemma we need is this.

Lemma. Suppose that T is just as strong as Q. If $\varphi(x, y)$ represents f as a function in T, then

$$\hat{\varphi}(x, y) =_{df} \{\varphi(x, y) \wedge \exists! y \varphi(x, y)\} \vee \{y = 0 \wedge \neg\exists! y \varphi(x, y)\}$$

fully represents f in T.

Proof. Suppose φ represents f in T. We need to show two things:

(1) $T \vdash \forall x \exists! y \hat{\varphi}(x, y)$
(2) if $f(n) = m$, then $T \vdash \hat{\varphi}(\bar{n}, \bar{m})$.

Take the latter first.

Suppose $f(n) = m$. Then we have $T \vdash \varphi(\bar{n}, \bar{m})$ as well as $T \vdash \exists! y \varphi(\bar{n}, y)$. Conjoined, this is the instantiations of the first disjunct of $\hat{\varphi}$. By addition we get the instantiation of $\hat{\varphi}$. Thus, $T \vdash \hat{\varphi}(\bar{n}, \bar{m})$.

For the first claim, it suffices to show $T \vdash \exists! y \hat{\varphi}(a, y)$ for an arbitrary individual a. Now, reasoning within T, either $\exists! y \varphi(a, y)$ or $\neg\exists! y \varphi(a, y)$. Proceed by separation of cases.

Taking the first case, it follows that $\exists y\hat{\varphi}(x,y)$. It remains to show uniqueness, which again follows from the supposition. Taking the second case, conjoin the supposition with $0=0$. Existentially generalize to $\exists y\,\{y=0 \wedge \neg\exists! y\varphi(a,y)\}$. Then $\exists y\hat{\varphi}(a,y)$ follows. Uniqueness follows since $\hat{\varphi}(a,b)$ entails $b=0$.

6. P.R. ADEQUACY OF Q

6.1. **Preliminaries.**

Defn. A theory T is (weakly) p.r. adequate just in case every p.r. function is representable in T as a function (as a relation).

If T is p.r. adequate, then all p.r. relations are representable in T, since their characteristic functions are representable in T.

PRA_0 is primitive recursive arithmetic w/o quantifiers. It is like BA, but with a new function symbol and axiom schemata added for each p.r. function. This is possible since the p.r. functions are effectively enumerable. Since it is quantifier free, PRA_0 is complete via an argument similar to the argument for the completeness of BA.

PRA is primitive recursive arithmetic, constructed like PRA_0 only with quantifiers. Some treatments use this as a replacement for Q. The disadvantage is that you don't get the strong results for the language of Q.

Defn.

- f is a Δ_0 function iff f can be defined by a Δ_0 wff.
- f is a Σ_1 function iff f can be defined by a Σ_1 wff.
- f is a Π_1 function iff f can be defined by a Π_1 wff.

Lemma. Any Σ_1 function is also Π_1.

Proof. Let f be Σ_1 and let $\varphi(x,y)$ be a Σ_1 wff that defines it. For the sake of argument we can take φ to be strictly Σ_1. Hence φ has the form $\exists v_1 \cdots \exists v_n \delta(x,y)$, where $\delta(x,y)$ is Δ_0. Now note that, that since f is a function, $\forall z(\varphi(x,z) \rightarrow z=y)$ also defines f, i.e. $\forall z(\exists v_1 \cdots \exists v_n \delta(x,y) \rightarrow z=y)$ also defines f. This is equivalent to $\forall z \forall v_1 \cdots \forall v_n(\delta(x,y) \rightarrow z=y)$, which is Π_1.

Lemma. Any Δ_0 function is representable in Q as a function by a Δ_0 wff.

Proof. Suppose f is Δ_0 function expressed by the Δ_0 wff $\varphi(x,y)$. Since Q correctly decides every Δ_0 sentence, for every m,n, if $f(n)=m$, then $Q \vdash \varphi(\bar{n},\bar{m})$, and if $f(n) \neq m$, then $Q \vdash \neg\varphi(\bar{n},\bar{m})$. Hence φ represents f as a relation in Q. By an earlier result, $\tilde{\varphi}(x,y) =_{df} \varphi(x,y) \wedge (\forall z \leq y)(\varphi(x,z) \rightarrow z=y)$ represents f in Q as a function. Since the universal quantification is bounded, this is still a Δ_0 wff.

6.2. **Every Σ_1 Function Is Representable in Q.**

Lemma. Any Σ_1 function is equivalent to the composition of two Δ_0 functions.

Proof. Let $f(x) = y$ be a Σ_1 function. Hence, there is a Δ_0 wff $\rho(x, y, z)$ such that $\exists z \rho(x, y, z)$ defines $f(x) = y$. Let R be the relation defined by ρ. Now define two other functions as follows.

$$g(x) = (\mu n)(\exists y \leq n)(\exists z \leq n)Rxyz$$

Given x, $g(x)$ finds the minimum cap n under which we need to look to find a y and a z such that $\langle x, y, x\rangle \in R$. Next

$$h(x, u) = (\mu y \leq u)(\exists z \leq u)Rxyz \text{ if such an } y \text{ exists, or else } 0.$$

As a result, plugging in $g(x)$ for u, $h(x, n)$ returns the least y under the ceiling n such that there exists a z such that $\langle x, y, x\rangle \in R$. This means that

$$f(x) = h(x, g(x)).$$

And the point of this redefinition of $f(x)$ is that both g and h are Δ_0 functions, defined respectively by the wffs

$$\gamma(x, u) =_{df} (\exists y \leq u)(\exists z \leq u)\rho(x, y, z) \wedge (\forall v \leq u)[v \neq u \rightarrow (\exists y' \leq v)(\exists z' \leq v)\rho(x, y', z')]$$

and

$$\eta(x, u, v) =_{df} [(\exists z \leq u)\rho(x, v, z) \wedge \neg(\exists y \leq v)(\exists y \leq u)(y \neq v \wedge \rho(x, y, z))] \vee [\neg(\exists z \leq u)Rxvz \wedge v = 0]$$

Lemma. The composition of two Δ_0 functions is representable in Q as a function.

Proof. Suppose $f(x) = h(x, g(x))$ and g and h are Δ_0 wffs defined by $\gamma(x, y)$ and $\eta(x, u, y)$, respectively. Then, by the lemma before last, $\tilde{\gamma}$ and $\tilde{\eta}$ represent g and h, respectively, in Q as functions. The claim now is that Σ_1 wff

$$\varphi(x, y) =_{df} \exists u(\tilde{\gamma}(x, u) \wedge \tilde{\eta}(x, u, y))$$

represents f in Q as a function. By an observation from the last section, it suffices to show that if $f(n) = m$, then $Q \vdash \forall z(\varphi(\bar{n}, z) \leftrightarrow z = \bar{m})$. So, suppose $f(n) = m$. Then for some k, $g(n) = k$ and $h(n, k) = n$. Since $\tilde{\gamma}$ and $\tilde{\eta}$ represent g and h, respectively as functions, we have both

$$Q \vdash \forall z(\tilde{\gamma}(\bar{n}, z) \leftrightarrow z = \bar{k})$$

and

$$Q \vdash \forall z(\tilde{\eta}(\bar{n}, \bar{k}, z) \leftrightarrow z = \bar{m}).$$

It is an exercise to show that these entail that

$$Q \vdash \forall z(\varphi(\bar{n}, z) \leftrightarrow z = \bar{m}).$$

Hence the Σ_1 wff φ represents f in Q as a function.

Theorem. Any Σ_1 function is representable in Q as a function.

Proof. From the last two lemmata. I.e., Any Σ_1 function can be defined as the composition of two Δ_0 functions, and the composition of any two Δ_0 functions is representable as a function in Q by a Σ_1 wff.

6.3. **All P.R. Functions Are Definable (in the standard model).**

6.3.1. *All initial p.r. functions are definable.*

- Successor function: $Sx = y$
- Zero function: For $Z(x) = y$ use $(x = x \wedge y = 0$
- Projection functions: e.g., for $I^3_2(x, y, z) = w$ use $x = x \wedge y = w \wedge z = z$

N.B. These are all Δ_0.

6.3.2. *The composition of definable functions is definable.* Let $f(x) = h(g(x))$ and let $\gamma(x, y)$ and $\eta(x, y)$ define g and h, respectively. Then $\exists u(g(x, u) \wedge h(u, y))$ defines f.

6.3.3. *β functions.* We want a way of coding up finite sequences $k_0, k_1, \ldots, k_n$. One way of doing this is to use a single number b to code the sequence via it's prime factorization of the first n primes, i.e.,

$$b = {\pi_0}^{k_0} \cdot {\pi_1}^{k_1} \cdots {\pi_n}^{k_n}.$$

Defn. A two-place β-function is a numerical function $\beta(c, i)$ such that for any sequence of numbers $k_0, k_1, \ldots, k_n$ there is a code number c such that for every $i \leq n$, $\beta(c, i) = k_i$.

Since we don't have exponentiation as a primitive, we can't use a a two-place code function and prime factorization. But we can use a three-place function.

Defn. A three-place β-function is a function of the form $\beta(c, d, i)$ such that for any finite sequence of natural numbers $k_0, k_1, \ldots, k_n$ there is a pair of code numbers c, d such that for every $i \leq n$, $\beta(c, d, i) = k_i$.

Let $rm(c, d)$ be the remainder of dividing c by d. Let D be a sequence of $n + 1$ relatively prime numbers $d_0, d_1, \ldots, d_n$. Let $Rm(c, D)$ be the sequence of remainders

$$rm(c, d_0), rm(c, d_1), \ldots, rm(c, d_n).$$

And let $\pi_D = d_0 \cdot d_1 \cdots d_n$.

Lemma: Chinese Remainder Theorem. For any sequence D of relatively prime numbers, the sequences $Rm(c, D)$ as c runs from 0 to $\pi_D - 1$ are all distinct from one another.

Proof. Suppose otherwise. Then there are numbers $0 \leq c_1 < c_2 < \pi_D$ such that $Rm(c_1, D) = Rm(c_2, D)$. Let c $= c_2 - c_1$. Trivially $c < \pi_D$. And trivially, if $rm(c_1, d) = rm(c_2, d)$, then $d \mid c$. Thus, since $Rm(c_1, D) = Rm(c_2, D)$, $d_i \mid c$ for each i. And since the d_i are relatively prime, it must be that $\pi_D \mid c$, contradicting the fact that $c < \pi_D$.

Defn. Gödel's β-function.

$$\beta(c, d, i) = rm(c, d(i + 1) + 1)$$

Lemma. For any sequence $k_0, k_1, \ldots, k_n$ there exists c, d such that $\beta(c, d, i) = k_i$.

Proof. Set s to the greatest of $n, k_0, k_1, \ldots, k_n$, and set $d = s!$. We claim first that for $0 \leq i \leq n$, the numbers $d_i = d(i+1) + 1$ are relatively prime. Suppose otherwise, i.e., for some $1 \leq j < k \leq n+1$, that $dj + 1$ and $dk + 1$ have a common prime factor p. Since any number up to s leaves a remainder of 1 when dividing $s!j + 1$, we have $s < p$. But also, since $p \mid (dj+1)$ and $p \mid (dk+1)$, $p \mid d(k-j)$. Now $p \not| j$ else $p \not| (dj+1)$. So $p \mid (k-j) < n < s$. Hence, $p < s$, giving us a contradiction.

Hence the d_i are relatively prime. So by the Chinese Remainder Theorem, we get every possible sequence of remainders as we run through $Rm(c, D)$ for $c = 0$ to $\pi_D - 1$. And one of these sequences must be $k_0, k_1, \ldots, k_n$, since each $k_i < s$ and hence a potential remainder on division by the corresponding d_i.

Now the Δ_0 wff

$$\beta(c, d, i, y) =_{df} (\exists u \leq c)[c = \{S(d \cdot Si) \cdot u\} + y \land y \leq (d \cdot Si)]$$

defines the Gödel β-function.

6.3.4. *Functions defined from definable functions by recursion are definable.*

We'll take as a warm up exercise the case of factorial. Recall the definition.

$$0! = 1; \quad (Sx)! = x! \cdot Sx$$

So, to find $x!$ we construct the sequence

$$0!, 1!, 2!, \ldots, x!$$

where we move from the u-th member of the sequence to the $(u+1)$-th by multiplying the $u - th$ by Su. Putting $x! = y$, the p.r. definition says

> There exists a sequence $k_0, k_1, \ldots, k_x$ such that $k_0 = 1$ and, if $u < x$, then $k_{Su} = k_u \cdot Su$, and $k_x = y$.

Given the β-function, we can re-express this as follows.

> There exists c and d such that $\beta(c, d, 0) = 1$ and, if $u < x$, then $\beta(c, d, Su) = \beta(c, d, u) \cdot Su$, and $\beta(c, d, x) = y$.

Now, expressing this in the language of arithmetic:

$$\exists c \exists d(\beta(c, d, 0, 1) \land (\forall u \leq x)(u \neq x \to \exists v \exists w(\beta(c, d, Su, v) \land \beta(c, d, u, w) \land v = w \cdot Su)) \land \beta(c, d, x, y)).$$

Now we give the general case. Let

$$f(x, 0) = g(x) \tag{1}$$

$$f(x, Sy) = h(x, y, f(x)). \tag{2}$$

Going step by step, as with the factorial function, this says

> There exists a sequence $k_0, k_1, \ldots, k_y$ such that $k_0 = g(x)$ and, if $u < y$, then $k_Su = h(x, y, k_u)$, and $k_y = z$.

Using the β-function,

> There exist c and d such that $\beta(c, d, 0) = g(x)$ and, if $u < y$, then $\beta(c, d, Su) = h(x, y, \beta(c, d, u))$, and $\beta(c, d, y) = z$.

And finally in the formal language, where $\gamma(x, y)$ defines g and $\eta(x, y, z, w)$ defines h:

$$\exists c \exists d((\exists k \beta(c, d, 0, k) \wedge \gamma(0, k)) \wedge (\forall u \leq y)(u \neq y \rightarrow \exists v \exists w(\beta(c, d, Su, v) \wedge \beta(c, d, u, w) \wedge \gamma(x, u, v, w))) \wedge \beta(c, d, x$$

Theorem Every p.r. function is Σ_1.

Proof. Need to argue that existential quantifiers can be moved out across bounded quantification.

7. ARITHMETIZATION OF SYNTAX

7.1. **Gödel Numbering.** The Gödel numbers g(s) for each symbol s is as given in the table. N.B. We would do well at this point to reduce the number of primitive logical symbols, treating the remainder as meta-linguistic abbreviations.

$\neg$	$\wedge$	$\vee$	$\rightarrow$	$\leftrightarrow$	$\forall$	$\exists$	$=$	(	)	0	S	$+$	$\cdot$	x	y	z	...
1	3	5	7	9	11	13	15	17	19	21	23	25	27	2	4	6	...

For any string $s_0 s_1 \cdots s_n$ the Gödel number is $\pi_0^{g(s_0)} \pi_1^{g(s_1)} \cdots \pi_n^{g(s_n)}$. For any sequence of strings, $e_0, \ldots, e_k$, the super Gödel number is $\pi_0^{g(e_0)} \cdots \pi_n^{g(e_n)}$.

7.2. **More p.r. functions.**

R7. Concatenation: $\ulcorner \varphi \urcorner \star \ulcorner \psi \urcorner$ is the Gödel number of the concatenation of φ with ψ i.e., $\ulcorner \varphi \psi \urcorner$. It is p.r. as follows. For any m and n

$$\begin{aligned} m \star n = (\mu x \leq B_{m,n)}(\forall i \leq len(m))(exp(x, i) = exp(m, i)) \quad \wedge \\ \forall j \leq len(j)(exp(x, len(m) + j) = exp(n, j)), \end{aligned}$$

where $B_{m,n}$ is large enough, say π_{m+n}^{m+n}.

R8. Numeral: The function $Num(n)$ whose value is the Gödel number of the standard numeral for n [i.e., $S^n 0$]. Use recursion:

$$\begin{aligned} Num(0) &= \ulcorner 0 \urcorner = 2^{21} \\ Num(Sn) &= \ulcorner S \urcorner \star Num(n) = 2^{23} \star Num(n). \end{aligned}$$

R9. The function $diag(n)$**:** Here we have to make up some ground. For any wff $\varphi(y)$, $\varphi(\ulcorner \varphi \urcorner)$ gives what intuitively is the diagonal of $\varphi(y)$. For technical reasons, though, we define it to be $\exists y(y = \ulcorner \varphi \urcorner \wedge \varphi(y))$. This is equivalent to $\varphi(\ulcorner \varphi \urcorner)$.

Now, we have the function $diag(n)$ whose value is the g.n. of the diagonalization of the wff whose g.n is n. Thus,

$$diag(n) = \ulcorner \exists y(y = \urcorner \star Num(n) \star \ulcorner \wedge \urcorner \star n \star \ulcorner) \urcorner.$$

R10. Variables: The property $Var(n)$ that holds when n is the g.n. of a variable is p.r. Note we treat the variable here as a string of length one. We have $Var(n)$ iff $(\exists m \leq n)(2^{2 \cdot m} = n)$.

R11 Term Building Sequence: We say that a sequence is a term building sequence just in case each expression in the sequence is a variable, is 0, or follows from preceding members by S, $+$, or $\cdot$ constructions. The relation $Termseq(m, n)$ holds between the g.n. m of a term building sequence and the g.n. n of the last term in the sequence.

We have $Termseq(m, n)$ iff $n = exp(m, len(m) - 1) \wedge (\forall i \leq len(m) - 1)\varphi(i, m)$ where $\varphi(i, m)$ is the disjunction

$$\begin{array}{rcl} exp(m,i) = \ulcorner 0 \urcorner & \vee & Var(exp(m,i))) \\ & \vee & (\exists j < i)(exp(m,i) = \ulcorner S \urcorner \star exp(m,j)) \\ & \vee & (\exists j < i)(\exists k < i)(exp(m,i) = \ulcorner (\urcorner \star exp(m,j) \star \ulcorner + \urcorner \star exp(m,k) \star \ulcorner) \urcorner) \\ & \vee & (\exists j < i)(\exists k < i)(exp(m,i) = \ulcorner (\urcorner \star exp(m,j) \star \ulcorner \cdot \urcorner \star exp(m,k) \star \ulcorner) \urcorner) \end{array}$$

R12 Term:

$$Term(n) =_{df} (\exists m \leq B_n) Termseq(m, n),$$

where B_n is suitably large, e.g., $(\pi_l^n)^l$, where $l = len(n)$. Why so? The term contains at most l symbols, and so the term sequence is bounded by l.

R13 Atom:

$$Atom(n) =_{df} (\exists j, k \leq n)(Term(j) \wedge Term(k) \wedge n = j \star \ulcorner = \urcorner \star k)$$

R14 Formula Building Sequence:
R15 Wff:
R16 Free Vble:
R17 Sentence:
R18 Universal Generalization:
R19 Axiom Type 1:
Rn Axiom:
Modus Ponens:
Proof:

8. Semantic Incompleteness Argument

Let $Gdl(m, n)$ be the relation that holds when m is the super g.n. for a PA proof of the diagonalization of wff with g.n. n. This is primitive recursive since $Gdl(m, n)$ iff $Prf(m, diag(n))$. (Its characteristic function is the composition of characteristic functions.) Hence it's expressible by some wff $\mathsf{Gdl}(\mathsf{x}, \mathsf{y})$. Now define the wff

$$\mathsf{U}(\mathsf{y}) =_{df} \forall x \neg \mathsf{Gdl}(\mathsf{x}, \mathsf{y})$$

Take the diagonal, of which is not quite

$$\mathsf{U}(\ulcorner \mathsf{U} \urcorner) = \forall \mathsf{x} \neg \mathsf{Gdl}(\mathsf{x}, \ulcorner \forall \mathsf{x} \neg \mathsf{Gdl}(\mathsf{x}, \mathsf{y}) \urcorner),$$

but rather equivalently

$$\exists \mathsf{y}(\mathsf{y} = \ulcorner \mathsf{U} \urcorner \wedge \mathsf{U}(\mathsf{y})),$$

or in other words,

$$\mathsf{G} =_{df} \exists \mathsf{y}(\mathsf{y} = \ulcorner \forall \mathsf{x} \neg \mathsf{GdI}(\mathsf{x}, \mathsf{y}) \urcorner \wedge \forall \mathsf{x} \neg \mathsf{GdI}(\mathsf{x}, \mathsf{y})).$$

Manifestly, G is true iff $PA \nvdash \mathsf{G}$. Manifestly, by inspection: G is true iff there is no x s.t. x is a proof of G.

Theorem. If PA is sound (satisfied by the standard model), then PA is incomplete.
Proof. Suppose PA is sound (satisfied by the standard model). If $PA \vdash \neg \mathsf{G}$, then, since PA is sound, $\neg \mathsf{G}$ is true. Since G says that there is no proof of G, it follows that there *is* a proof of G. So, $PA \vdash \mathsf{G}$. I.e., PA is inconsistent and hence unsound. Alternatively, if $PA \vdash \mathsf{G}$, then G is true. And, since G iff there is no proof of G, it follows that $PA \nvdash \mathsf{G}$.

Generalizing,
Theorem. Let T be any theory whose language includes zero, successor, addition, and multiplication (and a predicate for natural number if its intended domain is larger). If T is pr axiomatizable, can express all primitive recursive functions, and is true in the standard model, then T is incomplete.

9. Goldbach Type

9.1. **Preliminaries.** Results we need from before.

- Any Σ_1 function is also Π_1.
- Q can capture any Δ_0 function as a function.
- Any Σ_1 function is equivalent to the composition of two Δ_0 functions. (Proved from a long trick.)
- Q can capture any composition of Δ_0 functions.
- Hence, Q can capture any Σ_1 function em as a function.
- Primitive recursive functions are Σ_1. (Proved from the form of their expressibility.)
- Hence Q can capture any primitive recursive function *as a function*.

Defn. A sentence σ is of Goldbach type iff σ is equivalent to a universal generalization about a primitive recursive property or relation.

Claim: G is of Goldbach type.
Proof.: $\mathsf{GdI}(\mathsf{x}, \mathsf{y})$ expresses a p.r. relation and thus is Σ_1. $\mathsf{GdI}(\mathsf{x}, \ulcorner \mathsf{U} \urcorner)$ is then also Σ_1. $\neg \mathsf{GdI}(\mathsf{x}, \ulcorner \mathsf{U} \urcorner)$ is the negation of a Σ_1, and hence a Π_1 wff. Thus so is $\forall y \neg \mathsf{GdI}(\mathsf{x}, \ulcorner \mathsf{U} \urcorner)$, which is logically equivalent to G.

Theorem. Any sentence of Goldbach type is Π_1.

Proof: Any p.r. function is Σ_1, but any Σ_1 function is also Π_1. So, adding more universal quantifiers keeps it equivalent to the universal generalization of a Δ_0 wff.

10. Syntactic Argument for Incompleteness

10.1. **An easy result.** *Observation.* If PA is consistent, then $PA \nvdash \mathsf{G}$.
Proof: Suppose $PA \vdash \mathsf{G}$. $Gdl(x, y)$ is primitive recursive, and hence there is a wff $\mathsf{Gdl(x, y)}$ that represents it in PA. Since G is the diagonalization of $U(y)$, there is some m such that $Gdl(m, \ulcorner U \urcorner)$. Hence $PA \vdash \mathsf{Gdl}(\bar{\mathsf{m}}, \ulcorner \mathsf{U} \urcorner)$. This is inconsistent with $\forall \mathsf{y} \neg \mathsf{Gdl}(\mathsf{x}, \ulcorner \mathsf{U} \urcorner)$, which is just G. Q.E.D.

10.2. **ω-completeness, consistency.**

Defn. An arithmetic theory T is *ω-incomplete* iff for some wff $\varphi(x)$, $T \vdash \varphi(\bar{m})$ for each m but $T \nvdash \forall x \varphi(x)$.

Illustration. If PA is consistent, then PA is ω-incomplete.

Proof. $PA \vdash \neg \mathsf{Gdl}(\bar{\mathsf{m}}, \ulcorner \mathsf{U} \urcorner)$ for each m but PA doesn't prove the universal geralization $\forall \mathsf{y} \neg \mathsf{Gdl}(\mathsf{x}, \ulcorner \mathsf{U} \urcorner)$, which is just G, (i.e., $PA \nvdash \mathsf{G}$).

Defn. An arithmetic theory T is *ω-inconsistent* iff $T \vdash \varphi(\bar{m})$ for each m yet $T \vdash \exists x \neg \varphi(x)$.

N.B. T is ω-consistent iff not ω-inconsistent. More positively: T is ω-consistent iff, if $T \vdash \varphi(\bar{m})$ for each m, then $T \nvdash \neg \forall x \varphi(x)$. Or contra-positively, $T \nvdash \varphi(\bar{m})$ for some m if $T \vdash \neg \forall x \varphi(x)$.

Observation. If T is ω-inconsistent, then T is not satisfied by the standard model.

Proof. Convert the syntactic definition into a semantic argument.

Theorem. There is a sentence G of Goldbach type such that if PA is consistent, then $PA \nvdash G$, and, if PA is ω-consistent, then $PA \nvdash \neg G$.

Proof. We've done the first part before. So pick up on the second claim. Suppose $PA \vdash \exists x \mathsf{Gdl}(\mathsf{x}, \ulcorner \mathsf{U} \urcorner)$. (This is just $\neg G$.) Yet for all m, $PA \vdash \neg \mathsf{Gdl}(\bar{\mathsf{m}}, \ulcorner \mathsf{U} \urcorner)$.[1] That makes PA ω-inconsistent.

> **Gödel's 1st incompleteness theorem:** Assume that T is p.r. axiomatized and can represent every p.r. function. Then there is a Goldbach type sentence unprovable if T consistent and it's negation unprovable if T is ω-consistent.

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[1] This may be a little quick. The reasoning is this. Recall that G is true iff $PA \nvdash \mathsf{G}$. Now, if PA is ω-consistent, then it's consistent. Hence $PA \nvdash \mathsf{G}$. So, G is true. Thus, for every m, $Gdl(m, \ulcorner U \urcorner)$ is false. Since $\mathsf{Gdl(x, y)}$ represents $Gdl(x, y)$, $PA \vdash \neg \mathsf{Gdl}(\bar{\mathsf{m}}, \ulcorner \mathsf{U} \urcorner)$ for each m.

11. MODEST EXTENSIONS OF GÖDEL'S 1ST THEOREM

Local Definition. A *nice* theory is consistent, p.r. axiomatized, and extends Q.

Observation. Craig's Theorem shows that any any effectively enumerable, theory is p.r. axiomatziable (by any argument of bounded for-loops). Thus, the p.r. axiomatized condition can be weakened accordingly.

Theorem. The theory of the intended model of arithmetic is not axiomatizable.

Proof: Suppose it is axiomatizable. Then it is nice and sound, and hence incomplete.

Rosser's Trick. Construct a sentence R_T that indirectly says, if I am provably in T, then my negation is already provable in T (i.e., with a smaller Gödel number). We have the usual semantic result that if T is nice and sound, then neither R_T nor $\neg R_T$ is provable from T. For assume T nice and sound, and that R_T is provable. Then, according to what R_T indirectly asserts, if R_T is provable, then so is it's negation. Having assume the antecedent, we get the consequent, which leads to a contraction. Hence R_T is not provable. This renders the conditional true in virtue of a false antecedent. Hence R_T is true and it's negation false. Since T is sound, $T \nvdash \neg R_T$, either.

Definitions.

1. T is ω-consistent iff there is n wff $\varphi(x)$ such that $T \vdash \exists x \varphi(x)$ while, for each m, $T \vdash \neg\varphi(\bar{m})$.
2. T is 1-consistent iff there is no Δ_0 formula $\varphi(x)$ such that $T \vdash \exists\vec{x}\varphi(\vec{x})$ while, for each $\vec{m}$, $T \vdash \neg\varphi(\vec{m})$
3. T is Σ_1 sound iff for every Σ_1 sentence σ such that $T \vdash \sigma$, σ is true.

Kreisel's Observation. You need to assume only that PA is 1-consistent in order to show that $PA \vdash \neg G$.

Proof. Suppose (1) $PA \vdash \neg G$. Note that $\neg G$ is equivalent to $\exists \mathsf{x}\mathsf{Gdl}(\mathsf{x}, \ulcorner \mathsf{U} \urcorner)$, where $\mathsf{Gdl}(\mathsf{x}, \ulcorner \mathsf{U} \urcorner)$ is Σ_1. Hence it is equivalent to some $\exists\vec{z}\psi(x,\vec{z})$, where $\psi(x,\vec{z})$ is Δ_0. Now (2) $PA \vdash \neg\mathsf{Gdl}(\bar{\mathsf{m}}, \ulcorner \mathsf{U} \urcorner)$ for each and every m. Conditions (1) and (2) are thus

(1) $PA \vdash \exists x \exists\vec{z}\psi(x,\vec{z})$
(2) $PA \vdash \neg\exists\vec{z}\psi(\bar{m},\vec{z})$ for each and every m.

Now (2) implies (by logic alone) $PA \vdash \neg\psi(\bar{m},\vec{n})$ for any sequence of numbers. Invoke now the hypothesis that (3) PA is 1-consistent. (1) and (3) entail that PA is 1-inconsistent. Hence you can't have (1) and (3). So you can't have (1) and (2) either. Hence $PA \nvdash \neg G$.

Observation. Σ_1 soundness entails consistency.

Proof. Suppose T is inconsistent. Then anything follows, including a Σ_1 and its negation, which both can't be true.

Observation. Let T be nice. Then T is 1-consistent iff Σ_1 sound.

Proof. Suppose T is 1-consistent but not Σ_1 sound. Then for some Δ_0 wff $\varphi(x)$ the Σ_1 wff $\exists x\varphi(x)$ is false even though $T \vdash \exists x\varphi(x)$. Since $\exists x\varphi(x)$ is false, $\neg\varphi(\bar{m})$ is true for some m. Since T is nice, it extends Q and hence proves every true Δ_0 wff. Hence $T \vdash \neg\varphi(\bar{m})$, contradicting the assumption that T is 1-consistent.

Conversely, suppose T is Σ_1 sound but not 1-consistent. Then for some Δ_0 wff $\varphi(x)$, $T \vdash \exists x\varphi(x)$ but $T \vdash \neg\varphi(\bar{m})$ for each m. Since $\exists x\varphi(x)$ is Σ_1 and T is Σ_1 sound, $\exists x\varphi(x)$ is true. Hence, in the standard model $\varphi(\bar{m})$ is true for some m. But since T is nice, it extends Q, which correctly decides each Δ_0 sentence. Thus $T \vdash \varphi(\bar{m})$, rendering T inconsistent. But nice theories are consistent.

12. Provability Predicates

Result. Let T be a nice theory. Then if $T \vdash \varphi$, then $T \vdash \mathsf{Prov}_T(\ulcorner\varphi\urcorner)$.

Proof. Suppose $T \vdash \varphi$. Then for some m, $\mathit{Prf}(m, \ulcorner\varphi\urcorner)$. Thus, since T is nice (in particular, extends Q), $\mathsf{Prf}(\mathsf{x},\mathsf{y})$ represents $\mathit{Prf}(j,k)$ in T, and so $T \vdash \mathsf{Prf}(\bar{\mathsf{m}}, \ulcorner\varphi\urcorner)$. Hence $T \vdash \mathsf{Prov}_T(\ulcorner\varphi\urcorner)$.

Result. Conversely, if T is nice and ω-consistent, then $T \vdash \varphi$ if $T \vdash \mathsf{Prov}_T(\ulcorner\varphi\urcorner)$.

Proof. Suppose $T \vdash \mathsf{Prov}_T(\ulcorner\varphi\urcorner)$, i.e., $T \vdash \exists x\mathsf{Prf}(x, \ulcorner\varphi\urcorner)$. Now, since T is ω-consistent, this existential claim must have a witness, m. So $\mathit{Prf}(m, \ulcorner\varphi\urcorner)$, and hence $T \vdash \varphi$.

13. Tarski's Theorem

Defn An open L' wff $\mathsf{T}(x)$ is a *formal truth-predicate for* L iff for every L sentence σ, $T \vdash \mathsf{T}(\ulcorner\sigma\urcorner) \leftrightarrow \sigma$.