Philosophy 421 R. Rynasiewicz
Mathematical Logic II Spring 2022

Homework # 1 Solutions

Due Tuesday, February 22 (100 points, 10 points each)

1. Show that the set of wff's of a countable language is countable.

   Soln. Since the set of wffs is a subset of the set of all finite sequences of symbols, it suffices to show that the latter is countable. Let $\mathscr{S}$ be the set of all symbols, including logical symbols and parentheses. Because $\mathscr{S}$ is countable, there exists an injection $f : \mathscr{S} \to \mathbb{N}$. Thus, for each finite sequence $\langle s_1, \ldots, s_n \rangle$ on $\mathscr{S}$, let

   $$F(\langle s_1, \ldots, s_n \rangle) = p_1^{f(s_1)+1} \cdot p_n^{f(s_n)+1}.$$

   The claim now is that $F : \mathscr{S}^{<\omega} \to \mathbb{N}$ is 1-1, and thus that $\mathscr{S}^{<\omega}$ is countable. For suppose that $F(\langle s_1, \ldots, s_n \rangle) = F(\langle t_1, \ldots, t_m \rangle)$. Then

   $$p_1^{f(s_1)+1} \cdot p_n^{f(s_n)+1} = p_1^{f(t_1)+1} \cdot p_m^{f(t_m)+1}.$$

   Since each p_i is prime, it follows that $m = n$ and that, for each $1 \leq i \leq n$, $f(s_i) + 1 = f(t_i) + 1$, and thus, since f is injective, that $s_i = t_i$. Therefore $\langle s_1, \ldots, s_n \rangle = \langle t_1, \ldots, t_m \rangle$, as promised.

2. Show that the theory of dense linear orderings without endpoints is $\aleph_0$-categorical, i.e., that any two countable models are isomorphic.

   Soln. Let $(A, <_A)$ and $(B, <_B)$ be countable dense linear orderings without endpoints. Since A and B are each countably infinite, each can be enumerated $a_0, a_1, \ldots$ and $b_0, b_1, \ldots$, respectively, without redundancy. We forge an isomorphism by defining an order-preserving bijection f by recursion on the indices of the first sequence. In order to do so, it helps to introduce some auxiliary quantities. Let

   $$B_n^+ =_{df} \{f(a_m) \,|\, m \leq n \ \wedge \ a_n \leq_A a_m\}.$$

   These are the values (under f) of those elements up through a_n that are greater than or equal to a_n in the $<_A$ ordering. Similarly, let

   $$B_n^- =_{df} \{f(a_m) \,|\, m \leq n \ \wedge \ a_m \leq_A a_n\}.$$

   Now, to define $f : A \to B$, we set $f(a_0) = b_0$ and $f(a_{n+1}) = b_m$, where m is the least k s.t.

   $$\max B_n^- <_B b_k <_B \min B_n^+.$$

We need to establish that f is bijective and order-preserving.

To show that f is 1-1 and order-preserving, first note that B_n^+ and B_n^- are non-empty for all n since the two orderings have no endpoints. So, suppose that $a_i \neq a_j$. Without loss of generality we can assume that $i < j$. Thus, $j = n+1$ for some n. If $a_i <_A a_{n+1}$, then $a_i \in B_n^-$ and

$$f(a_i) \leq_B \max B_n^- <_B f(a_{n+1}).$$

Conversely, if $a_{n+1} <_A a_i$, then $a_i \in B_n^+$ and

$$f(a_{n+1}) <_B \min B_n^+ \leq_B f(a_i).$$

To show that f is surjective, we argue by induction on the index m of the elements of B. Clearly $b_0 \in \text{ran } f$ since $f(a_0) = b_0$. Now suppose that that $b_m \in \text{ran } f$ (inductive hypothesis). Then $b_m = f(a_n)$ for some n. Because $<_B$ is a dense ordering, the set

$$B^* =_{df} \{b_j \mid \max B_n^- <_B b_j <_B \min B_n^+\}.$$

is non-empty, and thus has a least element $b_k = f(a_{n+1})$. ■

3. In our discussion of non-standard models of arithmetic, we invoked compactness in claiming that, because every finite subset of $\Theta \cup \text{Th } \mathfrak{N}$ has a model, $\Theta \cup \text{Th } \mathfrak{N}$ also has a model $\mathfrak{A}$. But we also claimed that $|\mathfrak{A}|$ can be taken to be countable. Justify this latter claim.

   Soln. Because the language is countable, the Löwenheim-Skolem theorem entails that $\Theta \cup \text{Th } \mathfrak{N}$ has a countable model. ■

4. Show that Robinson's **Q** is a subtheory of **PA**.

   Soln. Since each axiom except S3 of **Q** is an axiom of **PA**, it suffices to derive S3 from **PA**. Now, S3 is

   $$\forall x(x \neq 0 \rightarrow \exists y : x = Sy).$$

   We'll prove this by induction on x. The base case $x = 0$ is satisfied trivially. So, suppose that x has the property in question. Clearly $x+1$ has an immediate predecessor, viz., x. Thus, $x+1$ has the property in question. ■

5. Show that for all $m, n \in \mathbb{N}$, CA $\vdash \overline{m}+\overline{n} = \overline{m+n}$. *Suggestion*: First take m to be an arbitrary natural number. Then do the induction on the parameter n.

   Soln. Let m be an arbitrarily chosen natural number. As suggested, we argue by induction on n. For the base case, it is an axiom of CA that $\overline{m} + \overline{0} = \overline{m}$. Trivially,

$\overline{m+0} = \overline{m}$. Thus, $\overline{m} + \overline{0} = \overline{m+0}$. For the inductive step, suppose that for arbitrary n that CA $\vdash \overline{m} + \overline{n} = \overline{m+n}$. We need to show that CA $\vdash \overline{m} + \overline{n+1} = \overline{m+(n+1)}$. It is an axiom of CA that $\overline{m} + \overline{n+1} = \overline{(m+n)+1}$, and trivially $(m+n)+1 = m+(n+1)$. ■

6. Show that for all $m, n \in \mathbb{N}$, CA $\vdash \overline{m} \bullet \overline{n} = \overline{m \cdot n}$.

   Soln. Again, let m be an arbitrarily chosen natural number. We argue by induction on n. For the base case, it is an axiom of CA that $\overline{m} \bullet \overline{0} = \overline{m}$, and trivially $m + 0 = m$ so that CA $\vdash \overline{m} \bullet \overline{0} = \overline{m+0}$. For the inductive step, let n be any natural number and suppose that CA $\vdash \overline{m} \bullet \overline{n} = \overline{m \cdot n}$. We need to show that CA $\vdash \overline{m} \bullet \overline{n+1} = \overline{m \cdot (n+1)}$. It is an axiom of CA that $\overline{m} \bullet \overline{n+1} = \overline{m \cdot n} + \overline{m}$. From the last problem, we have

$$\begin{aligned} \overline{m \cdot n} + \overline{m} &= \overline{m \cdot n + m} \\ &= \overline{m \cdot (n+1)}. \end{aligned}$$

   ■

7. Show that for any variable free term τ, CA $\vdash \tau = \overline{n}$, where n is the denotation of τ in $\mathfrak{N}$ (i.e., where $\overline{s}(\tau) = n$ for any mapping s of variables into $|\mathfrak{N}|$). *Suggestion*: Do induction on the shape of τ.

   Soln. The base case is $\tau = \overline{0}$ and trivially CA $\vdash \overline{0} = \overline{0}$. For the inductive step, suppose that the claim holds for both τ and τ', i.e., that CA $\vdash \tau = \overline{n}$ and CA $\vdash \tau' = \overline{n'}$. We need to show that the claim holds for $\tau + \overline{1}$, $\tau + \tau'$ and $\tau \bullet \tau'$. Now, $\overline{n} + \overline{1}$ denotes $n+1$. From an earlier exercise, CA $\vdash \overline{n} + \overline{1} = \overline{n+1}$. Thus, the successor case goes through. For arbitrary addition, the denotation of $\tau + \tau'$ is $n + n'$. Since by hypothesis CA $\vdash \tau = \overline{n}$ and CA $\vdash \tau' = \overline{n'}$, we have CA $\vdash \tau + \tau' = \overline{n} + \overline{n'}$. From earlier we have CA $\vdash \overline{n} + \overline{n'} = \overline{n+n'}$. Thus, CA $\vdash \tau + \tau' = \overline{n+n'}$, as desired. *Mutatis mutandis* for multiplication.

8. Show that for any quantifier free sentence σ in the language of arithmetic both

$$\text{CA} \vdash \sigma \Leftrightarrow \sigma \in \text{Th } \mathfrak{N}$$

   and

$$\text{CA} \vdash \neg\sigma \Leftrightarrow \sigma \notin \text{Th } \mathfrak{N}.$$

   Soln. The left to right direction in the first biconditional amounts to saying that CA is sound in the sense that whatever is provable from the axioms of CA is arithmetically true. In other words, $\mathfrak{N}$ models the axioms of CA, which is apparant. The left to right direction in the second biconditional follows from that of the first: Suppose

CA$\vdash \neg\sigma$. From left to right in the first conditional, it follows that $\neg\sigma \in \text{Th } \mathfrak{N}$ and hence that $\sigma \notin \text{Th } \mathfrak{N}$.

We prove the right to left directions of both biconditionals simultaneouly using induction on the shape of σ. Let Γ be the set of all quantifier free sentences, which is freely generated from the set of atomic sentences using negation and material conditional. Further, let Δ be that subset of quantifier free sentences σ s.t.: if $\sigma \in \text{Th } \mathfrak{N}$, then CA$\vdash \sigma$, and if $\sigma \notin \text{Th } \mathfrak{N}$, then CA$\vdash \neg\sigma$. We show that Δ is inductive in the indicated sense.

For the base case, consider an atomic sentence. This has the form of an equation $t_1 = t_2$. Let n_1 and n_2 be the denotations of t_1 and t_2, respectively. (I.e., for any mapping of variables to numbers $\bar{s}(t_1) = n_1$ and $\bar{s}(t_2) = n_2$ since t_1 and t_2 must be variable free.) Now, suppose $t_1 = t_2$ is an element of Th $\mathfrak{N}$. By the previous exercise, CA $\vdash t_1 = \overline{n_1}$ and CA $\vdash t_2 = \overline{n_2}$. By symmetry and transitivity of identity, it follows that CA $\vdash t_1 = t_2$.

Alternatively, suppose that $t_1 \neq t_2$ is not an element of Th $\mathfrak{N}$. Then $n_1 \neq n_2$. Furthermore CA $\vdash \overline{n_1} \neq \overline{n_2}$. To see this, suppose without loss of generality that $n_1 < n_2$ and $k = n_2 - n_1$. Now, working in the object language, suppose that $\overline{n_1} \neq \overline{n_2}$. Applying axiom schema S2 n_1 times yields $\overline{0} = \overline{k}$, which violates an instance of schema S1. So, CA $\vdash \overline{n_1} \neq \overline{n_2}$. Since, by the last exercise, CA $\vdash t_1 = \overline{n_1}$ and CA $\vdash t_2 = \overline{n_2}$, it follows that CA $\vdash t_1 \neq t_2$.

So much for the base case. For the inductive step, suppose that $\sigma, \tau \in \Delta$. We need to show that $\neg\sigma \in \Delta$ and $(\sigma \to \tau) \in \Delta$.

Suppose that $\neg\sigma \in \text{Th } \mathfrak{N}$. Then $\sigma \notin \text{Th } \mathfrak{N}$, and hence, by the inductive hypothesis, CA $\vdash \neg\sigma$. On the other hand, suppose that $\neg\sigma \notin \text{Th } \mathfrak{N}$. Then $\sigma \in \text{Th } \mathfrak{N}$, and hence, by the inductive hypothesis, CA $\vdash \sigma$. Thus, CA $\vdash \neg\neg\sigma$.

Now suppose that $(\sigma \to \tau) \in \text{Th } \mathfrak{N}$. Then either (i) $\sigma \notin \text{Th } \mathfrak{N}$ or (ii) $\tau \in \text{Th } \mathfrak{N}$. In case (i), CA $\vdash \neg\sigma$ and thus CA $\vdash (\sigma \to \tau)$. In case (ii), CA $\vdash \tau$ and so CA $\vdash (\sigma \to \tau)$. On the other hand, suppose that $(\sigma \to \tau) \notin \text{Th } \mathfrak{N}$. Then $\sigma \in \text{Th } \mathfrak{N}$ while $\tau \notin \text{Th } \mathfrak{N}$. By the inductive hypothesis, CA $\vdash \sigma$ and CA $\vdash \neg\tau$. Hence, CA $\vdash \neg(\sigma \to \tau)$. ■

9. Show that for any quantifier-free sentence σ, either PA $\vdash \sigma$ or PA $\vdash \neg\sigma$.

   Soln. Let σ be a quantifier free sentence. Either $\sigma \in \text{Th } \mathfrak{N}$ or $\sigma \notin \text{Th } \mathfrak{N}$. Thus, by the last exercise, either CA $\vdash \sigma$ or CA $\vdash \neg\sigma$. Since every instance of every axiom schema of CA is provable in PA, it follows that PA $\vdash \sigma$ or PA $\vdash \neg\sigma$. ■

10. Show that the set of quantifier free arithmetic truths is decidable.

    Soln. Let σ be a quantifier free sentence in the language of arithmetic. To determine whether or not σ is an arithmetic truth, effectively enumerate the set of all theorems of PA. Go through the list until you find either σ or $\neg\sigma$. Since PA $\vdash \sigma$ or PA $\vdash \neg\sigma$, one or the other will show up, and, assuming PA consistent, only one of the two will show up. Thus, if σ shows up, then σ is an arithmetic truth, and if $\neg\sigma$ shows up, then σ is not an arithmetic truth.