Philosophy 421 | R. Rynasiewicz
Mathematical Logic II | Spring 2022

Homework # 2

Due Tuesday, March 1 (60 points)

1. Let $\mathfrak{A}$ be a model of A_S. Show that there is a unique isomorphic embedding $\phi : \mathbb{N} \to |\mathfrak{A}|$ of $\mathfrak{N}_S$ into $\mathfrak{A}$ such that

$$\phi(n) = (\mathbf{S}^{\mathfrak{A}})^n(\mathbf{0}^{\mathfrak{A}}).$$

[10 points]

2. Show that this embedding need not be onto. [10 points]

3. Suppose that for every wff φ of the form

$$\exists x(\alpha_0 \land \cdots \land \alpha_n),$$

where $\alpha_0, \ldots, \alpha_n$ are either atomic or the negation of atomic wffs, there is a quantifier-free wff ψ such that

$$T \models (\varphi \leftrightarrow \psi).$$

Show that T admits elimination of quantifiers. *Suggestion*: Recall that every wff has an equivalent in prenex normal form. Use induction. You can rely on the lemma that, give the supposition, then for every wff of the form $\exists x\theta$, where θ is quantifier free, there is a ψ such that

$$T \models (\exists x\theta \leftrightarrow \psi).$$

[10 points]

4. Use the method given in Enderton to find for each of the following wffs φ a quantifier free wff ψ s.t.

$$\text{Th } \mathfrak{N}_S \models (\varphi \leftrightarrow \psi).$$

(a) $\exists x(Sx \neq S^2y \land S^3y \neq Sz \land x \neq z)$
(b) $\exists x(Sx = S^2y \land S^3y \neq Sz \land x \neq z)$
(c) $\exists x(Sx \neq Sz \to S^2x = y \to y = z)$
(d) $\forall x\forall y(Sx = Sy \to x = y)$

[20 points]

5. Using the same procedure for generating from φ, a quantifier free ψ in the quantifier elimination problem for Th $\mathfrak{N}_S$ considered as a complete theory, show that

$$A_S \models (\varphi \leftrightarrow \psi)$$

without assuming the completeness of Cn A_S, and thus without assuming AC. *Suggestion*: Consider an arbitrary model $\mathfrak{A}$ of A_S and an arbitrary $s : Vble \to |\mathfrak{A}|$ and show that

$$\models_{\mathfrak{A}} (\varphi \leftrightarrow \psi) \quad [s].$$

[10 points]