Introductory Material

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Mathematical Logic II

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Elementary Languages

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Def. An **elementary language** $\mathcal{L}$ is a set

$$\mathcal{L} = \mathsf{Pred}(\mathcal{L}) \cup \mathsf{Fn}(\mathcal{L}),$$

where

$$\mathsf{Pred}(\mathcal{L}) = \bigcup_{n \in \mathbb{N}} \mathsf{Pred}_{n+1}(\mathcal{L}) \quad \text{and} \quad \mathsf{Fn}(\mathcal{L}) = \bigcup_{n \in \mathbb{N}} \mathsf{Fn}_n(\mathcal{L}),$$

where $\mathsf{Pred}_{n+1}(\mathcal{L})$ is called the set of $(n+1)$-adic **predicate symbols** of $\mathcal{L}$ and Fn_n the set of n-place **function symbols** of $\mathcal{L}$.

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- if $\varphi, \psi \in$ WFF($\mathcal{L}$), then
  - $\neg\varphi \in$ WFF($\mathcal{L}$),
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But what does it mean for a variable to be free in a wff?

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- $FV(\forall x\varphi) = FV(\varphi) \setminus \{x\}$

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- $(\varphi \leftrightarrow \psi) := ((\varphi \to \psi) \wedge (\psi \to \varphi))$
- $\exists x\, \varphi := \neg\forall x \neg\varphi$
- $\exists! x\, \varphi(x) := \exists x(\varphi(x) \wedge \forall y(\varphi(y) \to y = x))$,
  where y is the first variable not in $\varphi(x)$, and $\varphi(y)$ is the result of uniformly substituting y for x wherever x is free in $\varphi(x)$.

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N.B. $Str(\mathcal{L})$ is a proper class.

Sequences on a Domain of Discourse

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We're used to thinking of a sequence on A as $a_0, a_1, \ldots, a_i, \ldots$, but this is a listing of the range of ϕ, i.e., $\phi(0), \phi(1), \ldots, \phi(i), \ldots$.

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Thus, $s : V \to |\mathfrak{A}|$ becomes a sequence if we think of s as "acting on" the indices of the variables in V.

Def. Let $d \in |\mathfrak{A}|$. Then $s_{x \mapsto d}$ is the sequence on $|\mathfrak{A}|$ that assigns d to x but otherwise agrees everywhere else with s, i.e., for any $y \in V$:

$$s_{x \mapsto d}(y) = \begin{cases} d & \text{if } y = x \\ s(y) & \text{otherwise.} \end{cases}$$

Denotations of Arbitrary Terms

A sequence $s : V \to |\mathfrak{A}|$ provides for each variable a temporary denotation from the domain of discourse. It can be extended uniquely to a mapping

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- $\bar{s}(ft_1 \cdots t_n) = f^{\mathfrak{A}}(\bar{s}(t_1), \ldots, \bar{s}(t_n))$ for each $f \in \mathrm{Fn}_n$ and $t_1, \ldots, t_n \in \mathrm{Term}(\mathcal{L})$, $n > 0$.

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$$\models_{\mathfrak{A}} \varphi \, [s]$$

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- $\models_{\mathfrak{A}}\ \forall x \varphi\ [s]$ iff $\models_{\mathfrak{A}}\ \varphi\ [s_{x \mapsto d}]$ for all $d \in |\mathfrak{A}|$.

Truth in a Structure, Models of Sets of Sentences

Lemma. Suppose that sequences s and s' agree on all variables free in a wff φ. Then $\models_{\mathfrak{A}} \varphi\ [s]$ iff $\models_{\mathfrak{A}} \varphi\ [s']$.

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Lemma. Suppose that sequences s and s' agree on all variables free in a wff φ. Then $\models_{\mathfrak{A}} \varphi\ [s]$ iff $\models_{\mathfrak{A}} \varphi\ [s']$.

Thus, if σ is a sentence, either $\models_{\mathfrak{A}} \sigma\ [s]$ for all $s : V \to |\mathfrak{A}|$, or else $\not\models_{\mathfrak{A}} \sigma\ [s]$ for all $s : V \to |\mathfrak{A}|$. This facilitates the following definition.

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Def. If σ is a sentence, then $\models_{\mathfrak{A}} \sigma$ iff $\models_{\mathfrak{A}} \sigma\ [s]$ for some $s : V \to |\mathfrak{A}|$, in which case we say that σ is **true** in $\mathfrak{A}$, or σ **satisfies** $\mathfrak{A}$, or $\mathfrak{A}$ is a **model** of σ. If $\not\models_{\mathfrak{A}} \sigma$, then we say that σ is **false** in $\mathfrak{A}$, or that $\mathfrak{A}$ is a **countermodel** of σ or a **counterexample** to σ.

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For any system of derivation $\mathcal{S}$:

- a proof is a finite syntactic object (e.g., sequence of wffs), and
- there is a decision procedure for determining whether a given syntactic construction is a proof of τ from Σ, as long as Σ is decidable, i.e., as long as there is an effective procedure for determining for an arbitrary σ whether $\sigma \in \Sigma$.

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Notation. $\text{Cn } \Sigma := \{\tau : \Sigma \models \tau\}$, i.e., $\text{Cn } \Sigma$ is the set of logical consequences of Σ.

Deductive Soundness and Completeness

Def. A system of derivation $\mathcal{S}$ is **deductively sound** iff for any elementary language $\mathcal{L}$, for any wff φ of $\mathcal{L}$ and any set Γ of sentences of $\mathcal{L}$, if $\Gamma \vdash_{\mathcal{S}} \varphi$, then $\Gamma \models \varphi$.

Defn. A system of derivation $\mathcal{S}$ is **deductively complete** iff for any elementary language $\mathcal{L}$, for any wff φ of $\mathcal{L}$ and any set Γ of sentences of $\mathcal{L}$, if $\Gamma \models \varphi$, then $\Gamma \vdash_{\mathcal{S}} \varphi$.

Theorem (Gödel 1930). There exists a sound and complete system of derivation.

Corollary (Compactness Theorem). $\Gamma \models \varphi$ iff there exists a finite $\Gamma_0 \subseteq \Gamma$ s.t. $\Gamma_0 \models \varphi$, or equivalently: a set of wffs is satisfiable iff every finite subset is satisfiable.

Enumerability Theorem

Def. A set X is **enumerable** iff there exists a function $f : \mathbb{N} \to X$ that is onto X, i.e., there is a sequence which covers all of X.

Def. X is **effectively enumerable** iff there exists an effective procedure for enumerating X.

Def. An elementary language $\mathcal{L}$ is "**reasonable**" iff
(i) $\mathcal{L}$ is enumerable and
(ii) both $\mathrm{Pred}_{n+1}(\mathcal{L})$ and $\mathrm{Fn}_n(\mathcal{L})$ are decidable for each $n \in \mathbb{N}$.

Enumerability Theorem. In a reasonable language, if Σ is decidable (or even just effectively enumerable), then $\mathrm{Cn}\ \Sigma$ is effectively enumerable.

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Notation. $\text{Th } \mathfrak{A} := \text{Th } \{\mathfrak{A}\}$.

Uncountable Sets

Cantor's Theorem. For any set X, there is no function $f : X \to \mathcal{P}(X)$ that is onto, where $\mathcal{P}(X)$ is the power set of X

Cor. $\mathcal{P}(\mathbb{N})$ is uncountable (non-enumerable).

Proposition. There exists a bijection $f : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$.

Scholium. By definition, the first infinite cardinal $\aleph_0$ is the cardinality of $\mathbb{N}$. $\aleph_1$ is the least ordinal of cardinality greater than $\aleph_0$. In general, $\aleph_{\alpha+1}$ is the least ordinal of cardinality greater than $\aleph_\alpha$. If λ is a limit ordinal, then $\aleph_\lambda = \bigcup_{\gamma<\lambda} \aleph_\gamma$.

Continuum Hypothesis. The cardinality of $\mathcal{P}(\mathbb{N})$ is $\aleph_1$.

Cardinality Theorems for Models

Def. A theory T is **categorical** iff any two models of T are isomorphic.

Def. Let λ be a cardinal. A theory T is λ-**categorical** iff any two models of T of cardinality λ are isomorphic. (The cardinality of a structure is the cardinality of its domain of discourse.)

Lowenheim-Skolem Theorem. Any satisfiable theory in a countable language has a countable model.

Lowenheim-Skolem-Tarski Theorem (LST). If $\mathcal{L}$ is countable and T has an infinite model, then T has a model of all infinite cardinalities.

Properties of Theories

Def. A theory T is **consistent** iff there is no sentence σ such that both $\sigma \in T$ and $\neg\sigma \in T$.

N.B. For any language $\mathcal{L}$, there is only one inconsistent theory in $\mathcal{L}$, viz., the set of all sentences of $\mathcal{L}$.

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Axiomatizable and Complete $\Longrightarrow$ Decidable

Particular Examples

1. Theory of Dense Linear Orderings w/o Endpoints

Let $\mathcal{L}_{\mathcal{R}} = \{R\}$, where $R \in \text{Pred}_2(\mathcal{L}_{\mathcal{R}})$. (Read R as "is less than".) Consider the following axioms Δ:

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Thus, $\mathfrak{A} \equiv \mathfrak{B}$. ■

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Corollary. The theory of dense linear orderings w/o endpoints is decidable.

Particular Examples of Theories (cont.)

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Let $\mathcal{L}_F = \{\mathbf{0}, \mathbf{1}, +, \cdot\}$, where

- $\mathbf{0}, \mathbf{1} \in \mathsf{Fn}_0(\mathcal{L}_F)$
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- Multiplication distributes over addition.

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$$\begin{array}{ccc} \mathbf{0} & \not\approx & \mathbf{1}+\mathbf{1} \\ \mathbf{0} & \not\approx & \mathbf{1}+\mathbf{1}+\mathbf{1} \\ \vdots & \vdots & \vdots \end{array}$$

Then $F_0 = \mathrm{Cn}\,(\Phi \cup \Psi)$.

(This is not a result, but just an example of the axiomatic method of presenting a theory. An algebraic field of characteristic 0 just is any structure satisfying these axioms and equations.)

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What *is* a result is that F_0 is not *finitely* axiomatizable.

But to show this, we first need a lemma about finite axiomatizability in general.

Lemma on Finite Axiomatizability

Lemma. Suppose T is finitely axiomatizable and Σ axiomatizes T. Then there is a finite $\Sigma_0 \subseteq \Sigma$ s.t. $\text{Cn}\ \Sigma_0 = T$.

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So,

$$\text{Cn } \tau \subseteq \text{Cn } \Sigma_0 \subseteq \text{Cn } \Sigma,$$

and thus

$$\text{Cn } \Sigma_0 = T.$$

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Therefore F_0 is not finitely axiomatizable. ■

Properties of F_0 (cont.)

F_0 is *not complete* since both the real and complex fields are models, and the sentence

$$\exists x \, (x \cdot x + 1 = 0)$$

is false in the former but true in the latter.

Particular Examples of Theories (cont.)

3. Algebraically Closed Fields of Characteristic 0

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We arrive at the theory $\overline{F_0}$ of algebraically closed fields of characteristic 0 by adding in all sentences asserting the existence of solutions to all polynomials with arbitrary coefficients, i.e., sentences of the form:

$$\forall a_0 \ldots a_n \exists x (a_0 \cdot x^n + \cdots + a_n \approx \mathbf{0}),$$

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Corollary. $\overline{F_0}$ is decidable.

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Particular Examples of Theories (cont.)

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Gödel's 1st Incompleteness Theorem. Th $\mathfrak{N}$ is not axiomatizable (and hence undecidable). Alternatively, any arithmetically true axiomatization is incomplete. (If $\Sigma \subseteq$ Th $\mathfrak{N}$ is decidable, then Cn Σ is incomplete.)

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We will get even stronger results, e.g., any "sufficiently strong" axiomatizable subtheory is not only incomplete, but also undecidable.

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This can be seen by the following, simple application of the Compactness Theorem.

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First, note that the $<$ relation on $\mathbb{N}$ is definable in $\mathfrak{N}$ by

$$\exists x(x \not\approx \mathbf{0} \ \wedge \ v_0 + x \approx v_1).$$

So, if t_1 and t_2 are terms of $\mathcal{L}_A$ (not containing the variable x), let $t_1 < t_2$ be a metalinguistic abbreviation for

$$\exists x(x \not\approx \mathbf{0} \ \wedge \ t_1 + x \approx t_2).$$

Non-Standard Models of Arithmetic (cont.)

Next, introduce some further metalinguistic notation as follows. Let

$$\begin{aligned} \mathbf{S}^0\mathbf{0} &= \mathbf{0} \\ \mathbf{S}^{n+1}\mathbf{0} &= \mathbf{S}\mathbf{S}^n\mathbf{0}. \end{aligned}$$

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Finally, add a new constant c to $\mathcal{L}_A$ and add to Th $\mathfrak{N}$ the sentences:

$$\Theta = \{\mathbf{0} < c, \mathbf{S0} < c, \mathbf{SS0} < c, \ldots, \mathbf{S}^n\mathbf{0} < c, \ldots\}$$

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Every finite subset Σ_0 of $\Theta \cup$ Th $\mathfrak{N}$ has a model, viz.,

$$\mathfrak{N}^+ = (\mathbb{N}, 0, c^{\mathfrak{N}^+}, S, +, \cdot),$$

where $c^{\mathfrak{N}^+}$ is any natural number larger the largest n s.t. $\mathbf{S}^n\mathbf{0} < c \in \Sigma_0$.

Non-Standard Models of Arithmetic (cont.)

So $\Theta \cup \mathrm{Th}\,\mathfrak{N}$ has a model

$$\mathfrak{A} = (|\mathfrak{A}|, 0, c^{\mathfrak{A}}, S^{\mathfrak{A}}, +^{\mathfrak{A}}, \cdot^{\mathfrak{A}}),$$

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Clearly, $\mathfrak{A}^*$ satisfies Th $\mathfrak{N}$, but $\mathfrak{A}^* \not\cong \mathfrak{N}$.

Summary of Examples of Theories

Theory	Axiom.?	F.A.?	Complete?	Decidable?
Dense Lin. Ord. w/o Endpts.	Yes	Yes	Yes	Yes
Fields of Characteristic 0	Yes	No	No	
Th $\mathfrak{C}$	Yes	No	Yes	Yes
Th $\mathfrak{N}$	No	No	Yes	No
Peano Arithmetic	Yes	No	No	No
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Some Incomplete but Decidable Theories

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$\mathcal{L}_{\Pi_n} = \{F_1, \dots, F_n\}$, where $F_i \in \mathrm{Pred}_1(\mathcal{L}_{\Pi_n})$ for $1 \leq i \leq n$

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III. Theory $\Phi_1$of a Single Unary Function

$\mathcal{L}_{\Phi_1} = \{f\}$, where $f \in \text{Fn}_1(\mathcal{L}_{\Phi_1})$

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Some Incomplete but Decidable Theories (cont.)

IV. Theory E of Equivalence Relations

$\mathcal{L}_E = \{R\}$, where $R \in \mathrm{Pred}_2(\mathcal{L}_E)$

$\mathbf{E} = \mathrm{Cn}\ \tau$, where τ is the conjunction of the following.

$$\forall x Rxx$$

$$\forall x \forall y (Rxy \rightarrow Ryx)$$

$$\forall x \forall y \forall z (Rxy \rightarrow Ryz \rightarrow Rxz)$$

Some Incomplete but Decidable Theories (cont.)

V. Theory $\mathcal{G}_A$ of Abelian Groups.

$\mathcal{L}_{\mathcal{G}_A} = \{\circ, e\}$, where $\circ \in \mathrm{Fn}_2(\mathcal{L}_{\mathcal{G}_A})$ and $e \in \mathrm{Fn}_0(\mathcal{L}_{\mathcal{G}_A})$

$\mathcal{G}_A = \mathrm{Cn}\ \gamma$, where γ is the conjunction of the following.

$$\forall x \forall y \forall z\ ((x \circ y) \circ z = x \circ (y \circ z))$$

$$\forall x\ (x \circ e = x = e \circ x)$$

$$\forall x \exists y\ (x \circ y = e = y \circ x)$$

$$\forall x \forall y\ (x \circ y = y \circ x)$$

Some Incomplete but Decidable Theories (cont.)

VI. The Theory B of Boolean Algebras.

$\mathcal{L}_B = \{\wedge, \vee, \neg, 0, 1\}$, where

- $\wedge, \vee \in \mathrm{Fn}_2(\mathcal{L}_B)$,
- $\neg \in \mathrm{Fn}_1(\mathcal{L}_B)$, and
- $0, 1 \in \mathrm{Fn}_0(\mathcal{L}_B)$.

$\mathbf{B} = \mathrm{Cn}\ \beta$, where β is the conjunction of the following.

- $\wedge$ and $\vee$ are both associative and commutative
- $\wedge$ and $\vee$ distribute over one another
- $\forall x \forall y\ (x \vee (x \wedge y) = x)$
- $\forall x \forall y\ (x \wedge (x \vee y) = x)$
- $\forall x (x \vee 0 = x)$
- $\forall x (x \wedge 1 = x)$
- $\forall x (x \vee \neg x = 1)$
- $\forall x (x \wedge \neg x = 0)$

The Problem of Consistency

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If we assume in the metalanguage also that there exist *inaccessible cardinals*, then we can produce a model for **ZF**. (*Aside*: Intuitively, an inaccessible cardinal is one that has no predecessor nor is the limit of any sequence of smaller cardinals. It strongly inaccessible if it is larger than the power set of any smaller cardinal.) But this invites in turn the question, is the object language theory

$$\mathbf{ZF} + \text{‘There exist an inaccessible cardinal.’}$$

consistent?

The Problem of Consistency (cont.)

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Gödel's 2nd Incompleteness Theorem:
PA $\vdash Con_{PA}$ iff **PA** is inconsistent.