Robinson's Q

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Axioms of Robinson's Q

Recall the axioms for Robinson's System Q:

S1. $\forall x\,(\mathbf{0} \neq \mathbf{S}x)$

S2. $\forall x \forall y(\mathbf{S}x = \mathbf{S}y \rightarrow x = y)$

S3. $\forall x(x \neq \mathbf{0} \rightarrow \exists y(x = \mathbf{S}y))$

A1. $\forall x\,(x + \mathbf{0} = x)$

A2. $\forall x \forall y\,(x + \mathbf{S}y = \mathbf{S}(x + y))$

M1. $\forall x\,(x \bullet \mathbf{0} = \mathbf{0})$

M2. $\forall x \forall y\,(x \bullet \mathbf{S}y = x \bullet y + x)$

The axioms of Calculator Arithmetic (CA) are the numerical instances of axioms S1,S2,A1,A2,M1,M2. So, all the theorems of CA are theorems of Q. In particular, for every $n \in \mathbb{N}$,

$$Q \vdash \mathbf{0} + \overline{n} = \overline{n}.$$

However, . . .

“Trivial” Incompleteness of Q

Theorem. If $\mathfrak{N}$ is a model of Q (thus, if Q is consistent), then $Q \nvdash \forall x(\mathbf{0} + x = x)$.

Proof. Suppose $\mathfrak{N}$ is a model of Q. We construct from $\mathfrak{N}$ a model $\mathfrak{A}$ of Q in which $\forall x(\mathbf{0} + x = x)$ is false. Let $|\mathfrak{A}| = \mathbb{N} \cup \{a, b\}$, where a, b are distinct entities other than natural numbers. Stipulate that $\mathbf{0}^{\mathfrak{A}}, \mathbf{S}^{\mathfrak{A}}, +^{\mathfrak{A}}, \bullet^{\mathfrak{A}}$, extend $\mathbf{0}^{\mathfrak{N}}, \mathbf{S}^{\mathfrak{N}}, +^{\mathfrak{N}}, \bullet^{\mathfrak{N}}$, repectively, but that a and b behave as follows.

$$\mathbf{S}^{\mathfrak{A}}(a) = a$$

$$\mathbf{S}^{\mathfrak{A}}(b) = b$$

(Check that S1–S3 remain satisfied.)

"Trivial" Incompleteness of Q (cont.)

Letting n vary over $\mathbb{N}$ and x over $|\mathfrak{A}|$, stipulate that:

$$a +^{\mathfrak{A}} n = a \text{ and } b +^{\mathfrak{A}} n = b$$

$$x +^{\mathfrak{A}} a = b \text{ and } x +^{\mathfrak{A}} b = a$$

$$a \bullet^{\mathfrak{A}} 0 = 0 \text{ and } b \bullet^{\mathfrak{A}} 0 = 0$$

$$a \bullet^{\mathfrak{A}} x = b \text{ and } b \bullet^{\mathfrak{A}} x = a, \text{ if } x \neq 0$$

$$n \bullet^{\mathfrak{A}} a = b \text{ and } n \bullet^{\mathfrak{A}} b = a$$

(Check that A1, A2, M1, M2 are satisfied.)

But $0 +^{\mathfrak{A}} a = b \neq a$! ■

So Why Q?

Question: If Q is so trivially incomplete, why study Q at all?

*Answer**: Because Q is "sufficiently strong" in the sense mentioned earlier. I.e., every decidable subset of $\mathbb{N}$ is representable [can be captured] in Q.

Then it follows, by Smith's 2nd preview theorem, that Q is undecidable (and, of course, then every axiomatic extension of Q must also be undecidable).

What we need, though, is some general theory of "decidability" — that will come much later. Will show in the short term that Q can represent [capture] all primitive recursive properties.

Representing $\leq$ in Q

Obviously, the wff

$$\exists v_2(v_2 + v_0 = v_1)$$

defines the relation

$$\{\langle m, n\rangle \mid m \leq n\}$$

in $\mathfrak{N}$.

Claim. The wff $\exists v_2(v_2 + v_0 = v_1)$ represents $\{\langle m, n\rangle \mid m \leq n\}$ in Q.

Proof. We need to show for every $m, n \in \mathbb{N}$:

1. $m \leq n$ iff $Q \vdash \exists z\,(z + \overline{m} = \overline{n})$
2. $m \nleq n$ iff $Q \vdash \neg\exists z\,(z + \overline{m} = \overline{n})$

Proof of Representability of $\leq$ in Q

For (1), suppose that $m \leq n$. Then there is a unique $k \in \mathbb{N}$, s.t. $k + m = n$. We know that

$$\mathrm{CA} \vdash \overline{k} + \overline{m} = \overline{n}$$

(where CA is calculator arithmetic). And any theorem of CA is a theorem of Q (since every axiom of CA is an instance of an axiom of Q). Thus,

$$\mathrm{Q} \vdash \overline{k} + \overline{m} = \overline{n}.$$

Then, by existential generalization,

$$\mathrm{Q} \vdash \exists z(z + \overline{m} = \overline{n}).$$

The converse is just a matter that $\mathfrak{N}$ is a model of Q.

Proof of Representability of $\leq$ in Q (cont.)

For (2), suppose that $m \not\leq n$, i.e., $m > n$. Now, *working in the object language*, suppose

$$\exists z(z + S^m 0 = S^n 0).$$

Let c be such a z, i.e.,

$$c + S^m 0 = S^n 0.$$

We have then

$$S(c + S^{m-1} 0) = S^n 0.$$

Thus,

$$c + S^{m-1} 0 = S^{n-1} 0.$$

Iterate this n times to get

$$c + S^{m-n} 0 = 0.$$

Proof of Representability of $\leq$ in Q (cont.)

Since $m > n$, we have

$$S(c + S^{m-n-1}0) = 0,$$

violating the axiom that 0 has no immediate predecessor. By reductio ad absurdum, we conclude

$$\neg \exists z(z + S^m 0 = S^n 0).$$

Thus,

$$\mathrm{Q} \vdash \neg \exists z(z + S^m 0 = S^n 0).$$

As for the converse, this is again a matter that $\mathfrak{N}$ is a model of Q. ■

Notation

Adopt the following notational abbreviations for the object language:

- $t_1 \leq t_2$ for $\exists x(x + t_1 = t_2)$, where x is the first variable in neither t_1 nor t_2
- $(\forall x \leq t)\varphi$ for $\forall x(x \leq t \rightarrow \varphi)$
- $(\exists x \leq t)\varphi$ for $\exists x(x \leq t \wedge \varphi)$

The expressions $(\forall x \leq t)$ and $(\exists x \leq t)$ are called *bounded quantifiers*.

"Order Adequacy" of Q

Def. Q is said to be **order adequate** in that the following hold.

1. $Q \vdash \forall x\,(0 \leq x)$
2. For each n, $Q \vdash \forall x\,(x = 0 \vee x = 1 \vee \ldots \vee x = \bar{n} \rightarrow x \leq \bar{n})$
3. For each n, $Q \vdash \forall x\,(x \leq \bar{n} \rightarrow x = 0 \vee x = 1 \vee \ldots \vee x = \bar{n})$
4. For each n, if $Q \vdash \varphi(0)$, $Q \vdash \varphi(1)$, $\ldots$, and $Q \vdash \varphi(\bar{n})$, then $Q \vdash (\forall x \leq \bar{n})\varphi(x)$
5. For each n, if $Q \vdash \varphi(0)$, or $Q \vdash \varphi(1)$, or $\ldots$, or $Q \vdash \varphi(\bar{n})$, then $Q \vdash (\exists x \leq \bar{n})\varphi(x)$.
6. For each n, $Q \vdash \forall x(\bar{n} \leq x \rightarrow \bar{n} \leq Sx)$
7. For each n, $Q \vdash \forall x(\bar{n} \leq x \rightarrow \bar{n} = x \vee S\bar{n} \leq x)$
8. For each n, $Q \vdash \forall x(x \leq \bar{n} \vee \bar{n} \leq x)$
9. For each $n > 0$, if $Q \vdash (\forall x \leq \overline{n-1})\varphi(x)$, then $Q \vdash (\forall x \leq \bar{n})(x \neq \bar{n} \rightarrow \varphi(x))$

The Class of Δ_0 Wffs

Defn. The class of Δ_0 wffs is the class of wffs that can be built up from equations and inequalities with connectives and bounded quantifiers. To be explicit,

- Any wff of the form $t_1 = t_2$ or $t_1 \leq t_2$ is in Δ_0.
- If $\varphi, \psi \in \Delta_0$, then so are $\neg\varphi$ and $(\varphi \rightarrow \psi)$.
- If $\varphi \in \Delta_0$ and t a numeral or variable distinct from x, then $(\forall x \leq t)\varphi$ and $(\exists x \leq t)\varphi$ are in Δ_0.

Decidability of Δ_0-Sentences

Claim*. The set of Δ_0-sentences true in $\mathfrak{N}$ is decidable.

Proof*. (**N.B.** This claim has nothing to do with Q.)

- From Calculator Arithmetic (CA), we know that the set of all true equations is decidable. To decide an inequality $\tau \leq \tau'$, first find n and m s.t. $CA \vdash \tau = \overline{n}$ and $CA \vdash \tau' = \overline{m}$. Then test whether $n \leq m$.
- Second, note that negations and conditionals are decidable if their components are decidable.
- Finally, note that
  - $(\forall x \leq \overline{n})\varphi(x)$ can be decided by deciding $\varphi(0) \wedge \cdots \wedge \varphi(\overline{n})$, and
  - $(\exists x \leq \overline{n})\varphi(x)$ can be decided by deciding $\varphi(0) \vee \cdots \vee \varphi(\overline{n})$.

Q is Δ_0-Complete

Theorem. For any Δ_0-sentence σ,

1. $\mathsf{Q} \vdash \sigma$ if $\sigma \in \text{Th}\,\mathfrak{N}$, and
2. $\mathsf{Q} \vdash \neg\sigma$ if $\sigma \notin \text{Th}\,\mathfrak{N}$.

Pf. We already know that Q correctly decides any equation, i.e.,

- if $t_1 = t_2 \in \text{Th}\,\mathfrak{N}$, then $\mathsf{Q} \vdash t_1 = t_2$, and
- if $t_1 \neq t_2 \in \text{Th}\,\mathfrak{N}$, then $\mathsf{Q} \vdash t_1 \neq t_2$.

Furthermore, from conditions (2) and (3) of the order adequacy of Q, any inequality $t_1 \leq t_2$ is correctly decided by deciding a disjunction of equations.

From here we argue inductively.

Q is Δ_0 Complete (cont.)

- Suppose Q correctly decides σ.

  If $\neg\sigma \in \text{Th } \mathfrak{N}$, then $\sigma \notin \text{Th } \mathfrak{N}$, and, by the inductive hypothesis, $\text{Q} \vdash \neg\sigma$.

  Alternatively, if $\neg\sigma \notin \text{Th } \mathfrak{N}$, then $\sigma \in \text{Th } \mathfrak{N}$, and $\text{Q} \vdash \sigma$, which means $\text{Q} \vdash \neg\neg\sigma$.
- By similar reasoning, if Q correctly decides both φ and ψ, then Q correctly decides $(\varphi \rightarrow \psi)$.

Q is Δ_0 Complete (cont.)

- Suppose that Q correctly decides $\varphi(\overline{n})$ for each n.

  If $\models_{\mathfrak{N}} (\forall x \leq \overline{k})\varphi(x)$, then $\mathrm{Q} \vdash \varphi(0), \ldots, \mathrm{Q} \vdash \varphi(k)$, and so by condition (4) of the order adequacy of Q, $\mathrm{Q} \vdash (\forall x \leq \overline{k})\varphi(x)$.

  On the other hand, if $\not\models_{\mathfrak{N}} (\forall x \leq \overline{k})\varphi(x)$, then $\models_{\mathfrak{N}} \neg\varphi(\overline{m})$ for some $m \leq k$, and hence by condition (5) of the order adequacy of Q, $\mathrm{Q} \vdash (\exists x \leq \overline{k})\neg\varphi(x)$. From the latter $\neg(\forall x \leq \overline{k})\varphi$ is provable.

Q is Δ_0 Complete (cont.)

- Similarly, assume again that Q correctly decides $\varphi(\overline{n})$ for each n.

  Suppose that $\models_{\mathfrak{N}} (\exists x \leq \overline{k})\varphi(x)$. Then $\models_{\mathfrak{N}} \varphi(\overline{m})$ for some $m \leq k$. By condition (5), $\mathrm{Q} \vdash (\exists x \leq \overline{k})\varphi(x)$.

  On the other hand, suppose that $\not\models_{\mathfrak{N}} (\exists x \leq \overline{k})\varphi(x)$. Then $\models_{\mathfrak{N}} \neg\varphi(\overline{m})$ for each $m \leq k$, and, by the inductive hypothesis, $\mathrm{Q} \vdash \neg\varphi(\overline{m})$ for each $m \leq k$. By condition (4), it follows that $\mathrm{Q} \vdash (\forall x \leq \overline{k})\neg\varphi(x))$. From this is provable $\neg(\exists x \leq \overline{k})\varphi(x)$.

■

The Class of Σ_1 Wffs

Defn. We define the class of Σ_1 wffs inductively:

- Any Δ_0 wff is Σ_1.
- If φ and ψ are Σ_1, then so are
  - $(\varphi \wedge \psi)$,
  - $(\varphi \vee \psi)$,
  - $(\forall x \leq t)\varphi$, and
  - $(\exists x \leq t)\varphi$.
- If $\varphi(x)$ is Σ_1, then so is $\exists x \varphi(x)$.

N.B. Σ_1 is **not** closed under negation. Else if $\varphi \in \Sigma_1$, then $\neg\exists x \neg\varphi \in \Sigma_1$, and Σ_1 would be closed under unrestricted universal quantification, completely changing the game. Σ_1 would in effect be the set of all wffs.

The Class of Π_1 Wffs

Defn. We also define the class of Π_1 wffs inductively:

- Any Δ_0 wff is Π_1.
- If φ and ψ are Π_1, then so are
  - $(\varphi \wedge \psi)$,
  - $(\varphi \vee \psi)$,
  - $(\forall x \leq t)\varphi$, and
  - $(\exists x \leq t)\varphi$.
- If $\varphi(x)$ is Π_1, then so is $\forall x \varphi(x)$.

N.B. Again, note that Π_1 is **not** closed under negation.

Some Examples

Let

$$\begin{array}{rcl}\psi(v_0) & := & (\exists x \leq v_0)(x + x = v_0) \\ \chi(v_0) & := & (v_0 \neq 1 \wedge (\forall x \leq v_0)(\forall y \leq v_0)(x \cdot y = v_0 \rightarrow x = 1 \ \vee \ y = 1)).\end{array}$$

Both wffs are Δ_0. $\psi(v_0)$ defines the set of even numbers, $\chi(v_0)$ the set of primes. The sentence

$$\exists x(\psi(x) \wedge \chi(x)),$$

which asserts there is an even prime, is Σ_1. Goldbach's conjecture,

$$\forall x(\psi(x) \wedge 3 \leq x \rightarrow (\exists y \leq x)(\exists z \leq x)(\chi(y) \wedge \chi(z) \wedge y + z = x)),$$

is Π_1. A Π_1 sentence is often said to be of **Goldbach type**.

The Effect of Negation

Lemma. The negation of a Δ_0 wff is a Δ_0 wff. The negation of a Σ_1 wff is equivalent to a Π_1 wff, and the negation of Π_1 wff is equivalent to a Σ_1 wff.

Pf. The first claim follows immediately from the definition of a Δ_0 wff. For the other two, argue by induction on the shape of φ taking advantage of the quantifier equivalences $\neg\exists x\varphi$ with $\forall x\neg\varphi$ and $\neg\forall x\varphi$ with $\exists x\neg\varphi$. ■

Σ_1/Π_1 Soundness and Completeness

Def. Let T be a theory in the language of arithmetic.

- T is Σ_1 **sound** if every Σ_1 sentence of T is true in $\mathfrak{N}$.
- T is Σ_1 **complete** if every Σ_1 sentence true in $\mathfrak{N}$ is a member of T.
- T is Π_1 **sound** if every Π_1 sentence of T is true in $\mathfrak{N}$.
- T is Π_1 **complete** if every Π_1 sentence true in $\mathfrak{N}$ is a member of T.

Example: Q is not Π_1 complete, since $\mathrm{Q} \nvdash \forall x(0 + x = x)$.

Q Is Σ_1 Complete

Theorem. Q is Σ_1 complete.

Pf. By induction on the degree of σ, where the degree of σ is the number of connectives, bound quantifiers, or unbound quantifiers. Suppose σ is true in $\mathfrak{N}$.

- If σ is of degree zero, then σ is atomic, and hence Δ_0.
- σ is a conjunction of Σ_1 sentences. Then each conjunct is true, and by the inductive hypothesis provable in Q. Thus σ is as well.
- σ is a disjunction of Σ_1 sentences. Then at least one is true, and thus by the inductive hypothesis provable in Q. So, σ is as well.

Q Is Σ_1 Complete (cont.)

- σ is of the form $(\forall x < \overline{n})\varphi(x)$. Then $\varphi(0), \ldots, \varphi(\overline{n})$ are all true, and, by the inductive hypothesis provable in Q. Thus, by condition (4) of the order adequacy of Q, $(\forall x < \overline{n})\varphi(x)$ is provable in Q.
- σ is of the form $(\exists x < \overline{n})\varphi(x)$. Then at least one of $\varphi(0), \ldots, \varphi(\overline{n})$ is true, and, by the inductive hypothesis, provable in Q. By condition (5) of the order adequacy of Q, $(\exists x < \overline{n})\varphi(x)$ is also provable in Q.
- σ is of the form $\exists x \varphi(x)$. Then $\varphi(\overline{n})$ is true for some $n \in \mathbb{N}$ and hence provable in Q. Then so is the existential generalization $\exists x \varphi(x)$. ■

Corollaries of the Σ_1 Completeness of Q

Cor. A Π_1 sentence σ is true in $\mathfrak{N}$ iff σ is consistent with Q (assuming Q consistent).

Pf. Suppose σ is consistent with Q. That means $Q \nvdash \neg\sigma$, where $\neg\sigma$ is equivalent to a Σ_1 sentence τ. Since Q is Σ_1 complete, τ is false and hence σ true in $\mathfrak{N}$. The converse is a matter that $\mathfrak{N}$ is a model of Q. ■

Cor. Suppose theory T is an extension of Q. Then T is consistent iff T is Π_1 sound.

Pf. Suppose T is inconsistent. Then every Π_1 sentence is a theorem of T, and hence T is not Π_1 sound. On the other hand, suppose T is not Π_1 sound. Let σ be a false Π_1 theorem of T. Then $\neg\sigma$ is equivalent to a true Σ_1 sentence τ. Hence, $Q \vdash \tau$ and so $T \vdash \tau$, rendering T inconsistent. ■