Introductory Material

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Mathematical Logic II

Spring 2022

Elementary Languages

Logical Symbols : $\{\forall, \approx, \neg, \rightarrow\}$

Individual Variables : $V = \{v_i \mid i \in \mathbb{N}\} = \{v_0, v_1, v_2, \ldots\}$

Punctuation : $\{(,)\}$

Def. An **elementary language** $\mathcal{L}$ is a set

$$\mathcal{L} = \mathrm{Pred}(\mathcal{L}) \cup \mathrm{Fn}(\mathcal{L}),$$

where

$$\mathrm{Pred}(\mathcal{L}) = \bigcup_{n \in \mathbb{N}} \mathrm{Pred}_{n+1}(\mathcal{L}) \text{ and } \mathrm{Fn}(\mathcal{L}) = \bigcup_{n \in \mathbb{N}} \mathrm{Fn}_n(\mathcal{L}),$$

where $\mathrm{Pred}_{n+1}(\mathcal{L})$ is called the set of $(n+1)$-adic **predicate symbols** of $\mathcal{L}$ and Fn_n the set of n-place **function symbols** of $\mathcal{L}$.

Elementary Languages (cont.)

Def. If $c \in \mathsf{Fn}_0(\mathcal{L})$, then c is said to be an **individual constant** of $\mathcal{L}$.

Def. $\mathsf{Term}(\mathcal{L})$, the set of **terms** of $\mathcal{L}$, is defined recursively:

- $x \in \mathsf{Term}(\mathcal{L})$ if $x \in V$
- $c \in \mathsf{Term}(\mathcal{L})$ if $c \in \mathsf{Fn}_0(\mathcal{L})$
- if $t_1, \ldots, t_n \in \mathsf{Term}(\mathcal{L})$ and $f \in \mathsf{Fn}_n$, then $ft_1 \cdots t_n \in \mathsf{Term}(\mathcal{L})$

Elementary Languages (cont.)

Def. WFF($\mathcal{L}$), the set of **well-formed formulae** of $\mathcal{L}$, is defined recursively:

- if $t, t' \in \text{Term}(\mathcal{L})$, then $\approx tt' \in \text{WFF}(\mathcal{L})$
- if $t_1, \ldots, t_n \in \text{Term}(\mathcal{L})$ and $P \in \text{Pred}_n(\mathcal{L})$, then $Pt_1 \cdots t_n \in \text{WFF}(\mathcal{L})$
- if $\varphi, \psi \in \text{WFF}(\mathcal{L})$, then
  - $\neg\varphi \in \text{WFF}(\mathcal{L})$,
  - $(\varphi \to \psi) \in \text{WFF}(\mathcal{L})$, and
  - $\forall x \varphi \in \text{WFF}(\mathcal{L})$ for any $x \in V$

Def. A wff α is said to be **atomic** iff neither $\neg$, nor $\to$, nor $\forall$ occur in α.

Def. $\sigma \in \text{WFF}(\mathcal{L})$ is said to be a **sentence** of $\mathcal{L}$ if no variable is free in σ.

But what does it mean for a variable to be free in a wff?

Elementary Languages (cont.)

Def. Let $\alpha \in \mathrm{WFF}(\mathcal{L})$. $FV(\alpha)$, the set of variables **free** in α, is defined recursively:

- For any $x \in V$, if α is atomic, then $x \in FV(\alpha)$ iff x occurs in α.
- $FV(\neg\varphi) = FV(\varphi)$
- $FV((\varphi \to \psi)) = FV(\varphi) \cup FV(\psi)$
- $FV(\forall x\varphi) = FV(\varphi) \setminus \{x\}$

Defined Symbols

In the following, the left hand side is a meta-linguistic abbreviation for the object language wff on the right.

- $(\varphi \vee \psi) := (\neg\varphi \to \psi)$
- $(\varphi \wedge \psi) := \neg(\varphi \to \neg\psi)$
- $(\varphi \leftrightarrow \psi) := ((\varphi \to \psi) \wedge (\psi \to \varphi))$
- $\exists x\, \varphi := \neg\forall x \neg\varphi$
- $\exists! x\, \varphi(x) := \exists x(\varphi(x) \wedge \forall y(\varphi(y) \to y = x))$, where y is the first variable not in $\varphi(x)$, and $\varphi(y)$ is the result of uniformly substituting y for x wherever x is free in $\varphi(x)$.

Structures for Elementary Languages

Def. A **structure** $\mathfrak{A}$ for an elementary language $\mathcal{L}$ is a function s.t.

- $\mathfrak{A}(\forall) := |\mathfrak{A}|$ is a non-empty set called the domain of discourse
- if $P \in \mathrm{Pred}_n(\mathcal{L})$, then $\mathfrak{A}(P) := P^{\mathfrak{A}} \subseteq |\mathfrak{A}|^n$
- if $c \in \mathrm{Fn}_0(\mathcal{L})$, then $\mathfrak{A}(c) \in |\mathfrak{A}|$. ($\mathfrak{A}(c)$ is standardly denoted as $c^{\mathfrak{A}}$.)
- if $f \in \mathrm{Fn}_n(\mathcal{L})$ for $n > 0$, then $\mathfrak{A}(f) := f^{\mathfrak{A}}$ is an n-ary operation $f^{\mathfrak{A}} : |\mathfrak{A}|^n \to |\mathfrak{A}|$

Bastard Notation: Suppose $\mathcal{L} = \{P, c, f\}$. Then

$$\mathfrak{A} = (|\mathfrak{A}|, c^{\mathfrak{A}}, f^{\mathfrak{A}}, P^{\mathfrak{A}}).$$

Def. $Str(\mathcal{L}) =_{df} \{\mathfrak{A} \mid \mathfrak{A} \text{ is a structure for } \mathcal{L}\}$.

N.B. $Str(\mathcal{L})$ is a proper class.

Sequences on a Domain of Discourse

Let $\mathfrak{A} \in Str(\mathcal{L})$ and $s : V \to |\mathfrak{A}|$. We say that s is a **sequence** on $\mathfrak{A}$ on the following grounds.

Def. A(n infinite) **sequence** on a set A is a mapping $\phi : \mathbb{N} \to A$.

We're used to thinking of a sequence on A as $a_0, a_1, \ldots, a_i, \ldots$, but this is a listing of the range of ϕ, i.e., $\phi(0), \phi(1), \ldots, \phi(i), \ldots$.

Thus, $s : V \to |\mathfrak{A}|$ becomes a sequence if we think of s as "acting on" the indices of the variables in V.

Def. Let $d \in |\mathfrak{A}|$. Then $s_{x \mapsto d}$ is the sequence on $|\mathfrak{A}|$ that assigns d to x but otherwise agrees everywhere else with s, i.e., for any $y \in V$:

$$s_{x \mapsto d}(y) = \begin{cases} d & \text{if } y = x \\ s(y) & \text{otherwise.} \end{cases}$$

Denotations of Arbitrary Terms

A sequence $s : V \to |\mathfrak{A}|$ provides for each variable a temporary denotation from the domain of discourse. It can be extended uniquely to a mapping

$$\bar{s} : \mathrm{Term}(\mathcal{L}) \to |\mathfrak{A}|$$

thereby providing for each term of the language $\mathcal{L}$ a denotation in $|\mathfrak{A}|$ as follows.

- $\bar{s}(x) = s(x)$ for each $x \in V$.
- $\bar{s}(c) = c^{\mathfrak{A}}$ for each $c \in \mathrm{Fn}_0(\mathcal{L})$.
- $\bar{s}(ft_1 \cdots t_n) = f^{\mathfrak{A}}(\bar{s}(t_1), \ldots, \bar{s}(t_n))$ for each $f \in \mathrm{Fn}_n$ and $t_1, \ldots, t_n \in \mathrm{Term}(\mathcal{L})$, $n > 0$.

Satisfaction Conditions

The expression

$$\models_{\mathfrak{A}} \varphi\ [s]$$

means that $\mathfrak{A}$ **satisfies** wff φ with/under sequence s. This is defined recursively:

- $\models_{\mathfrak{A}} \approx t_1 t_2\ [s]$ iff $\bar{s}(t_1) = \bar{s}(t_2)$.
- $\models_{\mathfrak{A}}\ P t_1 \cdots t_n\ [s]$ iff $\langle \bar{s}(t_1), \ldots, \bar{s}(t_n) \rangle \in P^{\mathfrak{A}}$.
- $\models_{\mathfrak{A}}\ \neg\varphi\ [s]$ iff $\not\models_{\mathfrak{A}}\ \varphi\ [s]$.
- $\models_{\mathfrak{A}}\ (\varphi \to \psi)\ [s]$ iff $\not\models_{\mathfrak{A}}\ \varphi\ [s]$ or $\models_{\mathfrak{A}}\ \psi\ [s]$ (or both).
- $\models_{\mathfrak{A}}\ \forall x \varphi\ [s]$ iff $\models_{\mathfrak{A}}\ \varphi\ [s_{x \mapsto d}]$ for all $d \in |\mathfrak{A}|$.

Truth in a Structure, Models of Sets of Sentences

Lemma. Suppose that sequences s and s' agree on all variables free in a wff φ. Then $\models_{\mathfrak{A}} \varphi\ [s]$ iff $\models_{\mathfrak{A}} \varphi\ [s']$.

Thus, if σ is a sentence, either $\models_{\mathfrak{A}} \sigma\ [s]$ for all $s : V \to |\mathfrak{A}|$, or else $\not\models_{\mathfrak{A}} \sigma\ [s]$ for all $s : V \to |\mathfrak{A}|$. This facilitates the following definition.

Def. If σ is a sentence, then $\models_{\mathfrak{A}} \sigma$ iff $\models_{\mathfrak{A}} \sigma\ [s]$ for some $s : V \to |\mathfrak{A}|$, in which case we say that σ is **true** in $\mathfrak{A}$, or σ **satisfies** $\mathfrak{A}$, or $\mathfrak{A}$ is a **model** of σ. If $\not\models_{\mathfrak{A}} \sigma$, then we say that σ is **false** in $\mathfrak{A}$, or that $\mathfrak{A}$ is a **countermodel** of σ or a **counterexample** to σ.

Def. Let Σ be a set of sentences. Then $\mathfrak{A}$ is a model of Σ iff $\mathfrak{A}$ is a model of τ, for each $\tau \in \Sigma$.

Notation. Mod $\Sigma := \{\mathfrak{A} \in Str(\mathcal{L}) \mid \mathfrak{A}$ is a model of $\Sigma\}$.

Notation. Mod $\tau :=$ Mod $\{\tau\}$.

Logical Consequence *vs.* Provability

Def. τ is a **logical consequence** of Σ, written $\Sigma \models \tau$, iff $\mathrm{Mod}\ \Sigma \subseteq \mathrm{Mod}\ \tau$.

This is to be distinguished from the notion $\Sigma \vdash_{\mathcal{S}} \tau$, viz., that τ is **provable** from Σ in a given system of derivation $\mathcal{S}$.

For any system of derivation $\mathcal{S}$:

- a proof is a finite syntactic object (e.g., sequence of wffs), and
- there is a decision procedure for determining whether a given syntactic construction is a proof of τ from Σ, as long as Σ is decidable, i.e., as long as there is an effective procedure for determining for an arbitrary σ whether $\sigma \in \Sigma$.

Notation. $\mathrm{Cn}\ \Sigma := \{\tau : \Sigma \models \tau\}$, i.e., $\mathrm{Cn}\ \Sigma$ is the set of logical consequences of Σ.

Deductive Soundness and Completeness

Def. A system of derivation $\mathcal{S}$ is **deductively sound** iff for any elementary language $\mathcal{L}$, for any wff φ of $\mathcal{L}$ and any set Γ of sentences of $\mathcal{L}$, if $\Gamma \vdash_{\mathcal{S}} \varphi$, then $\Gamma \models \varphi$.

Defn. A system of derivation $\mathcal{S}$ is **deductively complete** iff for any elementary language $\mathcal{L}$, for any wff φ of $\mathcal{L}$ and any set Γ of sentences of $\mathcal{L}$, if $\Gamma \models \varphi$, then $\Gamma \vdash_{\mathcal{S}} \varphi$.

Theorem (Gödel 1930). There exists a sound and complete system of derivation.

Corollary (Compactness Theorem). $\Gamma \models \varphi$ iff there exists a finite $\Gamma_0 \subseteq \Gamma$ s.t. $\Gamma_0 \models \varphi$, or equivalently: a set of wffs is satisfiable iff every finite subset is satisfiable.

Enumerability Theorem

Def. A set X is **enumerable** iff there exists a function $f : \mathbb{N} \rightarrow X$ that is onto X, i.e., there is a sequence which covers all of X.

Def. X is **effectively enumerable** iff there exists an effective procedure for enumerating X.

Def. An elementary language $\mathcal{L}$ is "**reasonable**" iff
(i) $\mathcal{L}$ is enumerable and
(ii) both $\mathrm{Pred}_{n+1}(\mathcal{L})$ and $\mathrm{Fn}_n(\mathcal{L})$ are decidable for each $n \in \mathbb{N}$.

Enumerability Theorem. In a reasonable language, if Σ is decidable (or even just effectively enumerable), then $\mathrm{Cn}\ \Sigma$ is effectively enumerable.

Theories

Let T be a set of sentences in some elementary language $\mathcal{L}$.

Def. T is a **theory** iff $T = \mathrm{Cn}\ T$.

One has two ways of producing theories:

1. **Axiomatic Method**: Take $T = \mathrm{Cn}\ \Sigma$ for some *decidable* set Σ of sentences.
2. **Semantic Method**: Take $T = \mathrm{Th}\ \mathcal{K}$, where
    - $\mathcal{K} \subseteq \mathit{Str}(\mathcal{L})$, and
    - $\mathrm{Th}\ \mathcal{K} := \{\sigma \in \mathit{Sent}(\mathcal{L}) : \models_{\mathfrak{A}} \sigma \text{ for all } \mathfrak{A} \in \mathcal{K}\}$.

Notation. $\mathrm{Th}\ \mathfrak{A} := \mathrm{Th}\ \{\mathfrak{A}\}$.

Uncountable Sets

Cantor's Theorem. For any set X, there is no function $f : X \to \mathscr{P}(X)$ that is onto, where $\mathscr{P}(X)$ is the power set of X

Cor. $\mathscr{P}(\mathbb{N})$ is uncountable (non-enumerable).

Proposition. There exists a bijection $f : \mathscr{P}(\mathbb{N}) \to \mathbb{R}$.

Scholium. By definition, the first infinite cardinal $\aleph_0$ is the cardinality of $\mathbb{N}$. $\aleph_1$ is the least ordinal of cardinality greater than $\aleph_0$. In general, $\aleph_{\alpha+1}$ is the least ordinal of cardinality greater than $\aleph_\alpha$. If λ is a limit ordinal, then $\aleph_\lambda = \bigcup_{\gamma<\lambda} \aleph_\gamma$.

Continuum Hypothesis. The cardinality of $\mathscr{P}(\mathbb{N})$ is $\aleph_1$.

Cardinality Theorems for Models

Def. A theory T is **categorical** iff any two models of T are isomorphic.

Def. Let λ be a cardinal. A theory T is λ-**categorical** iff any two models of T of cardinality λ are isomorphic. (The cardinality of a structure is the cardinality of its domain of discourse.)

Lowenheim-Skolem Theorem. Any satisfiable theory in a countable language has a countable model.

Lowenheim-Skolem-Tarski Theorem (LST). If $\mathcal{L}$ is countable and T has an infinite model, then T has a model of all infinite cardinalities.

Properties of Theories

Def. A theory T is **consistent** iff there is no sentence σ such that both $\sigma \in T$ and $\neg\sigma \in T$.

N.B. For any language $\mathcal{L}$, there is only one inconsistent theory in $\mathcal{L}$, viz., the set of all sentences of $\mathcal{L}$.

Def. T is **axiomatizable** iff there is a decidable set Σ of sentences s.t. $T = \mathrm{Cn}\ \Sigma$.

Def. T is **finitely axiomatizable** iff $T = \mathrm{Cn}\ \sigma$ for some sentence σ.

Def. T is **complete** iff for every sentence σ, either $\sigma \in T$ or $\neg\sigma \in T$.

Def. T is **decidable** iff T is a decidable set of sentences.

Relations Amongst Properties of Theories

Assuming we are considering theories in a “reasonable” language, then:

Finitely Axiomatizable $\Longrightarrow$ Axiomatizable

Axiomatizable $\Longleftrightarrow$ Effectively Enumerable

Decidable $\Longrightarrow$ Effectively Enumerable

Axiomatizable and Complete $\Longrightarrow$ Decidable

Particular Examples

1. Theory of Dense Linear Orderings w/o Endpoints

Let $\mathcal{L}_{\mathcal{R}} = \{R\}$, where $R \in \mathrm{Pred}_2(\mathcal{L}_{\mathcal{R}})$. (Read R as "is less than".)
Consider the following axioms Δ:

$\forall x \forall y \forall z (xRy \rightarrow (yRz \rightarrow xRz))$ (transitivity)

$\forall x \forall y (xRy \rightarrow \neg yRx)$ (asymmetry)

$\forall x \forall y (xRy \vee yRx \vee x = y)$ (trichotomy)

$\forall x \forall y (xRy \rightarrow \exists z (xRz \wedge zRy))$ (density)

$\forall x \exists y \exists z (yRx \wedge xRz)$ (no endpoints)

Cn Δ is finitely axiomatizable, complete, and hence decidable.

Furthermore Cn $\Delta = \mathrm{Th}\ (\mathbb{Q}, <) = \mathrm{Th}\ (\mathbb{R}, <)$.

Łoś–Vaught Test (1954)

How do we know that Cn Δ is complete?

Łoś–Vaught Test. Let T be a theory in a countable language.
If
(i) T has no finite models, and
(ii) T is λ-categorical for some transfinite cardinal λ,
then T is complete.

Proof. First note that if any two models $\mathfrak{A}, \mathfrak{B}$ of a theory T are elementarily equivalent ($\mathfrak{A} \equiv \mathfrak{B}$), then T is complete.

(The reason is that then, since $T = \text{Th Mod } T$, we have $T = \text{Th } \mathfrak{A}$, and the latter is complete.)

So it suffices to show that any two models of T are elementarily equivalent.

Łoś–Vaught Test (1954), *cont.*

Let $\mathfrak{A}$ and $\mathfrak{B}$ be models of T. By LST there exist $\mathfrak{A}', \mathfrak{B}'$ of cardinality λ s.t. $\mathfrak{A} \equiv \mathfrak{A}'$ and $\mathfrak{B} \equiv \mathfrak{B}'$.

By hypothesis, T is λ-categorical. Hence

$$\mathfrak{A} \equiv \mathfrak{A}' \simeq \mathfrak{B}' \equiv \mathfrak{B}.$$

Thus, $\mathfrak{A} \equiv \mathfrak{B}$. ■

Completeness of Cn Δ

How does the Łoś–Vaught Test apply to the theory of dense linear orderings w/o endpoints?

Lemma (Cantor). Any two countable dense linear orderings w/o endpoints are isomorphic.

Proof. *Exercise*.

Thus, Cn Δ is $\aleph_0$-categorical, and hence, by the Łoś–Vaught Test, complete.

Corollary. The theory of dense linear orderings w/o endpoints is decidable.

Particular Examples of Theories (cont.)

2. The Theory F_0 of Algebraic Fields of Characteristic Zero

Let $\mathcal{L}_F = \{\mathbf{0}, \mathbf{1}, +, \cdot\}$, where

- $\mathbf{0}, \mathbf{1} \in \mathsf{Fn}_0(\mathcal{L}_F)$
- $+, \cdot \in \mathsf{Fn}_2(\mathcal{L}_F)$.

Let Φ be the (finite) set of axioms for an algebraic field:

- Addition and multiplication are both associative and commutative.
- $\mathbf{0}$ and $\mathbf{1}$ are additive and multiplicative identity elements, respectively.
- Everything has an additive inverse.
- Everything but $\mathbf{0}$ has a multiplicative inverse.
- Multiplication distributes over addition.

The Theory F_0 of Fields of Characteristic Zero (cont.)

Let Ψ be the infinite set of sentences:

$$\begin{array}{ccc} \mathbf{0} & \not\approx & \mathbf{1}+\mathbf{1} \\ \mathbf{0} & \not\approx & \mathbf{1}+\mathbf{1}+\mathbf{1} \\ \vdots & \vdots & \vdots \end{array}$$

Then $F_0 = \mathrm{Cn}\,(\Phi \cup \Psi)$.

(This is not a result, but just an example of the axiomatic method of presenting a theory. An algebraic field of characteristic 0 just is any structure satisfying these axioms and equations.)

What *is* a result is that F_0 is not *finitely* axiomatizable.

But to show this, we first need a lemma about finite axiomatizability in general.

Lemma on Finite Axiomatizability

Lemma. Suppose T is finitely axiomatizable and Σ axiomatizes T. Then there is a finite $\Sigma_0 \subseteq \Sigma$ s.t. $\mathrm{Cn}\ \Sigma_0 = T$.

Proof. Since T is finitely axiomatizable and since Σ axiomatizes T, there exists a sentence τ s.t.

$$\mathrm{Cn}\ \tau = T = \mathrm{Cn}\ \Sigma.$$

Hence, $\Sigma \models \tau$, and, by the compactness theorem, there exists a finite $\Sigma_0 \subseteq \Sigma$ s.t. $\Sigma_0 \models \tau$.

So,

$$\mathrm{Cn}\ \tau \subseteq \mathrm{Cn}\ \Sigma_0 \subseteq \mathrm{Cn}\ \Sigma,$$

and thus

$$\mathrm{Cn}\ \Sigma_0 = T.$$

■

Application of the Lemma to F_0

Corollary. F_0 is not finitely axiomatizable.

Proof. Suppose F_0 is finitely axiomatizable. Then $\mathrm{Cn}\ \Sigma_0 = F_0$ for some finite $\Sigma_0 \subset \Phi \cup \Psi$.

It must be the case that Σ_0 contains the field axioms Φ, else Σ_0 will have models that are not algebraic fields.

Hence $\Sigma_0 = \Phi \cup \Psi_0$ for some finite $\Psi_0 \subset \Psi$.

But then Σ_0 admits fields of finite characteristic.

Therefore F_0 is not finitely axiomatizable. ■

Properties of F_0 (cont.)

F_0 is **not complete** since both the real and complex fields are models, and the sentence

$$\exists x\,(x{\cdot}x + 1 = 0)$$

is false in the former but true in the latter.

Particular Examples of Theories (cont.)

3. Algebraically Closed Fields of Characteristic 0

We arrive at the theory $\overline{F_0}$ of algebraically closed fields of characteristic 0 by adding in all sentences asserting the existence of solutions to all polynomials with arbitrary coefficients, i.e., sentences of the form:

$$\forall a_0 \ldots a_n \exists x (a_0 \cdot x^n + \cdots + a_n \approx \mathbf{0}),$$

where x^n is short for $x \cdot \ \cdots \ \cdot x$ with x iterated n times.

Theorem. $\overline{F_0}$ is complete.

Proof. $\overline{F_0}$ has no finite models, and by some deeper facts of algebra, $\overline{F_0}$ is λ-categorical in all uncountable cardinals λ. So, completeness follows from the Łoś–Vaught Test. ■

Corollary. $\overline{F_0}$ is decidable.

Particular Examples of Theories (cont.)

4. Theory of the Field of Complex Numbers

Let $\mathfrak{C}$ be a structure for $\mathcal{L}_F$ such that

- $\mathfrak{C}(\forall) = \mathbb{C}$
- $\mathbf{0}^{\mathfrak{C}} = 0$
- $\mathbf{1}^{\mathfrak{C}} = 1$
- $\boldsymbol{+}^{\mathfrak{C}} = +$ on $\mathbb{C}$
- $\boldsymbol{\cdot}^{\mathfrak{C}} = \cdot$ on $\mathbb{C}$

Then $\mathfrak{C} = (\mathbb{C}, 0, 1, +, \cdot)$ is the complex field.

Theorem. Th $\mathfrak{C} = \overline{F_0}$

Proof. $\mathfrak{C} \in \mathrm{Mod}\ \overline{F_0}$ and $\overline{F_0}$ is complete. ■

Particular Examples of Theories (cont.)

5. Arithmetic

Let $\mathcal{L}_A = \{\mathbf{0}, \mathbf{S}, +, \cdot\}$, where

- $\mathbf{0} \in \mathsf{Fn}_0(\mathcal{L}_A)$
- $\mathbf{S} \in \mathsf{Fn}_1(\mathcal{L}_A)$
- $+, \cdot \in \mathsf{Fn}_2(\mathcal{L}_A)$.

Let $\mathfrak{N}$ be the structure for $\mathcal{L}_A$ such that

- $\mathfrak{N}(\forall) = \mathbb{N}$
- $\mathbf{0}^{\mathfrak{N}} = 0$
- $\mathbf{S}^{\mathfrak{N}} = S : \mathbb{N} \to \mathbb{N}$ s.t. $S(n) = n + 1$
- $+^{\mathfrak{N}} = +$ on $\mathbb{N}$
- $\cdot^{\mathfrak{N}} = \cdot$ on $\mathbb{N}$

Arithmetic (cont.)

I.e., $\mathfrak{N} = (\mathbb{N}, 0, S, +, \cdot)$ is the standard structure of arithmetic.

Th $\mathfrak{N}$ is the set of all arithmetic truths.

Gödel's 1st Incompleteness Theorem. Th $\mathfrak{N}$ is not axiomatizable (and hence undecidable). Alternatively, any arithmetically true axiomatization is incomplete. (If $\Sigma \subseteq$ Th $\mathfrak{N}$ is decidable, then Cn Σ is incomplete.)

We will get even stronger results, e.g., any "sufficiently strong" axiomatizable subtheory is not only incomplete, but also undecidable.

Non-Standard Models of Arithmetic

LST $\Longrightarrow$ Th $\mathfrak{N}$ has uncountable models.

So Th $\mathfrak{N}$ has non-standard models, i.e., models not isomorphic to $\mathfrak{N}$.

Are any two *countable* models of Th $\mathfrak{N}$ isomorphic? I.e., is Th $\mathfrak{N}$ $\aleph_0$-categorial? Answer is: NO.

This can be seen by the following, simple application of the Compactness Theorem.

First, note that the $<$ relation on $\mathbb{N}$ is definable in $\mathfrak{N}$ by

$$\exists x(x \not\approx \mathbf{0} \ \wedge \ v_0 + x \approx v_1).$$

So, if t_1 and t_2 are terms of $\mathcal{L}_A$ (not containing the variable x), let $t_1 < t_2$ be a metalinguistic abbreviation for

$$\exists x(x \not\approx \mathbf{0} \ \wedge \ t_1 + x \approx t_2).$$

Non-Standard Models of Arithmetic (cont.)

Next, introduce some further metalinguistic notation as follows. Let

$$\begin{aligned} \mathbf{S}^0\mathbf{0} &= \mathbf{0} \\ \mathbf{S}^{n+1}\mathbf{0} &= \mathbf{S}\mathbf{S}^n\mathbf{0}. \end{aligned}$$

Finally, add a new constant c to $\mathcal{L}_A$ and add to Th $\mathfrak{N}$ the sentences:

$$\Theta = \{\mathbf{0} < c, \mathbf{S0} < c, \mathbf{SS0} < c, \ldots, \mathbf{S}^n\mathbf{0} < c, \ldots\}$$

Every finite subset Σ_0 of $\Theta \cup \text{Th } \mathfrak{N}$ has a model, viz.,

$$\mathfrak{N}^+ = (\mathbb{N}, 0, c^{\mathfrak{N}^+}, S, +, \cdot),$$

where $c^{\mathfrak{N}^+}$ is any natural number larger the largest n s.t. $\mathbf{S}^n\mathbf{0} < c \in \Sigma_0$.

Non-Standard Models of Arithmetic (cont.)

So $\Theta \cup \mathrm{Th}\ \mathfrak{N}$ has a model

$$\mathfrak{A} = (|\mathfrak{A}|, 0, c^{\mathfrak{A}}, S^{\mathfrak{A}}, +^{\mathfrak{A}}, \cdot^{\mathfrak{A}}),$$

where $|\mathfrak{A}|$ is countable.

Delete $c^{\mathfrak{A}}$ from $\mathfrak{A}$, calling the result $\mathfrak{A}^*$, i.e.,

$$\mathfrak{A}^* = (|\mathfrak{A}|, 0, S^{\mathfrak{A}}, +^{\mathfrak{A}}, \cdot^{\mathfrak{A}}).$$

Clearly, $\mathfrak{A}^*$ satisfies Th $\mathfrak{N}$, but $\mathfrak{A}^* \not\cong \mathfrak{N}$.

Summary of Examples of Theories

Theory	Axiom.?	F.A.?	Complete?	Decidable?
Dense Lin. Ord. w/o Endpts.	Yes	Yes	Yes	Yes
Fields of Characteristic 0	Yes	No	No	
Th $\mathfrak{C}$	Yes	No	Yes	Yes
Th $\mathfrak{N}$	No	No	Yes	No
Peano Arithmetic	Yes	No	No	No
Robinson's $\mathbf{Q}$	Yes	Yes	No	No

But what are these last two theories?

Axioms of Robinson's **Q**

- $\forall x: \; Sx \neq 0$
- $\forall x, y (Sx = Sy \rightarrow x = y)$
- $\forall x (x \neq 0 \rightarrow \exists y : x = Sy)$
- $\forall x: \; x + 0 = x$
- $\forall x, y: \; x + Sy = S(x + y)$
- $\forall x: \; x \cdot 0 = 0$
- $\forall x, y: \; x \cdot Sy = x \cdot y + x$

Axioms of Peano Arithmetic

- $\forall x : \; Sx \neq 0$
- $\forall x, y (Sx = Sy \rightarrow x = y)$
- $\forall x : \; x + 0 = x$
- $\forall x, y : \; x + Sy = S(x + y)$
- $\forall x : \; x \cdot 0 = 0$
- $\forall x, y : \; x \cdot Sy = x \cdot y + x$
- Induction Schema

For any wff $\varphi(x; w_1, \ldots, w_n)$ in which y is not free, the universal closure of the following is an axiom.

$$(\varphi(0; w_1, \ldots, w_n) \land \forall x(\varphi(x) \rightarrow \varphi(Sx))) \rightarrow \forall x \varphi(x; w_1, \ldots, w_n)$$

Some Incomplete but Decidable Theories

Are there any examples of incomplete axiomatizable theories that are decidable?

Yes, inter alia . . .

I. Theory I of Identity

$\mathcal{L}_I = \emptyset$

$\mathbf{I} = \mathrm{Cn}\ \emptyset = \mathrm{Th}\ Str(\mathcal{L}_I)$

II. Theory Π_n of n-Many Monadic Properties

$\mathcal{L}_{\Pi_n} = \{F_1, \ldots, F_n\}$, where $F_i \in \mathrm{Pred}_1(\mathcal{L}_{\Pi_n})$ for $1 \leq i \leq n$

$\Pi_n = \mathrm{Cn}\ \emptyset = \mathrm{Th}\ Str(\mathcal{L}_{\Phi_n})$

III. Theory $\Phi_1$of a Single Unary Function

$\mathcal{L}_{\Phi_1} = \{f\}$, where $f \in \mathrm{Fn}_1(\mathcal{L}_{\Phi_1})$

$\Phi_1 = \mathrm{Cn}\ \emptyset = \mathrm{Th}\ Str(\mathcal{L}_{\Phi_1})$

Some Incomplete but Decidable Theories (cont.)

IV. Theory E of Equivalence Relations

$\mathcal{L}_E = \{R\}$, where $R \in \mathrm{Pred}_2(\mathcal{L}_E)$

$\mathbf{E} = \mathrm{Cn}\ \tau$, where τ is the conjunction of the following.

$$\forall x Rxx$$

$$\forall x \forall y (Rxy \to Ryx)$$

$$\forall x \forall y \forall z (Rxy \to Ryz \to Rxz)$$

Some Incomplete but Decidable Theories (cont.)

V. Theory $\mathcal{G}_A$ of Abelian Groups.

$\mathcal{L}_{\mathcal{G}_A} = \{\circ, e\}$, where $\circ \in \mathrm{Fn}_2(\mathcal{L}_{\mathcal{G}_A})$ and $e \in \mathrm{Fn}_0(\mathcal{L}_{\mathcal{G}_A})$

$\mathcal{G}_A = \mathrm{Cn}\ \gamma$, where γ is the conjunction of the following.

$$\forall x \forall y \forall z\, ((x \circ y) \circ z = x \circ (y \circ z))$$

$$\forall x\, (x \circ e = x = e \circ x)$$

$$\forall x \exists y\, (x \circ y = e = y \circ x)$$

$$\forall x \forall y\, (x \circ y = y \circ x)$$

Some Incomplete but Decidable Theories (cont.)

VI. The Theory B of Boolean Algebras.

$\mathcal{L}_B = \{\wedge, \vee, \neg, 0, 1\}$, where

- $\wedge, \vee \in \mathsf{Fn}_2(\mathcal{L}_B)$,
- $\neg \in \mathsf{Fn}_1(\mathcal{L}_B)$, and
- $0, 1 \in \mathsf{Fn}_0(\mathcal{L}_B)$.

$\mathbf{B} = \mathsf{Cn}\ \beta$, where β is the conjunction of the following.

- $\wedge$ and $\vee$ are both associative and commutative
- $\wedge$ and $\vee$ distribute over one another
- $\forall x \forall y\ (x \vee (x \wedge y) = x)$
- $\forall x \forall y\ (x \wedge (x \vee y) = x)$
- $\forall x (x \vee 0 = x)$
- $\forall x (x \wedge 1 = x)$
- $\forall x (x \vee \neg x = 1)$
- $\forall x (x \wedge \neg x = 0)$

The Problem of Consistency

The above theories (1)–(5) and (I–VI) have been presumed consistent, since each has a model, and soundness tells us any satisfiable set of sentences is consistent.

But the existence of the models is secured by working in **ZF** in the meta-language. We can formalize **ZF** and then ask whether **ZF** consistent.

If we assume in the metalanguage also that there exist *inaccessible cardinals*, then we can produce a model for **ZF**. (*Aside*: Intuitively, an inaccessible cardinal is one that has no predecessor nor is the limit of any sequence of smaller cardinals. It strongly inaccessible if it is larger than the power set of any smaller cardinal.) But this invites in turn the question, is the object language theory

ZF + 'There exist an inaccessible cardinal.'

consistent?

The Problem of Consistency (cont.)

Could we, working in just in **ZF**, show that **ZF** is consistent? Gödel asked just this question, only in terms of Peano Arithmetic (**PA**).

The idea is to code in number theory the syntax of $\mathcal{L}_A$, including the property of provability from the axioms of **PA**, so that

$$\mathbf{PA} \vdash Prov_{PA}(\ulcorner\varphi\urcorner) \text{ iff } \mathbf{PA} \vdash \varphi,$$

where $\ulcorner\varphi\urcorner$ is the number ("Gödel number") that codes φ.

Then then there is a sentence of the object language, Con_{PA}, expressing the consistency of **PA**, viz., $\neg Prov_{PA}(\ulcorner 0 = 1\urcorner)$.

Gödel's 2nd Incompleteness Theorem:
PA $\vdash Con_{PA}$ iff **PA** is inconsistent.