Some Preview Theorems on Incompleteness

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Definability in a Structure

Notation (from Enderton). Let $\mathcal{L}$ be an elementary language, $\mathfrak{A}$ a structure for $\mathcal{L}$, and φ a wff of $\mathcal{L}$, such that $FV(\varphi) \subseteq \{v_0, \ldots, v_k\}$. Then we say that $\mathfrak{A}$ satisfies φ with $a_0, \ldots, a_k \in |\mathfrak{A}|$, written

$$\models_{\mathfrak{A}} \varphi \; [\![a_0, \ldots, a_k]\!],$$

if $\models_{\mathfrak{A}} \varphi \, [s]$ for some (and hence all) $s : V \to |\mathfrak{A}|$ s.t. $s(v_0) = a_0, \ldots, s(v_k) = a_k$.

Defn. The $(k+1)$-ary **relation defined by φ in** $\mathfrak{A}$ is

$$\{\langle a_0, \ldots, a_k \rangle \in |\mathfrak{A}|^{k+1} : \models_{\mathfrak{A}} \varphi \; [\![a_0, \ldots, a_k]\!]\}.$$

Definability in a Structure (cont.)

Defn. Let R be a $(k+1)$-ary relation on $|\mathfrak{A}|$. R is **definable in** $\mathfrak{A}$ iff there exists a wff φ with $FV(\varphi) \subseteq \{v_0, \ldots, v_k\}$ s.t.

$$R = \{\langle a_0, \ldots, a_k\rangle : \models_{\mathfrak{A}} \varphi \; [\![a_0, \ldots, a_k]\!]\}.$$

Example. The binary relation $\{\langle m, n\rangle \mid m \leq n\}$ on $\mathbb{N}$ is definable in $\mathfrak{N}$ by means of the wff

$$\exists v_2 \; v_0 \mathbf{+} v_2 = v_1.$$

N.B. The locution 'φ expresses R' in Smith's dialect just means that φ defines R in $\mathfrak{N}$.

Also, 'true' in Smith's dialect means 'true in $\mathfrak{N}$' in ours.

Definable Arithmetic Relations

Theorem. "Most" arithmetic relations are undefinable (inexpressible).

Proof. There are only countably many wff's, while the number of arithmetic relations is uncountable by Cantor's theorem (to be reviewed). ■

Scholium. Consider just the case of arithmetic unary relations.

A unary arithmetic relation is just a subset of $\mathbb{N}$.

There are uncountably many subsets of $\mathbb{N}$, but only countably many wffs in $\mathcal{L}_A$.

That leaves uncountably many undefinable unary arithmetic relations.

Definable Arithmetic Relations (cont.)

Scholium (cont). For the *n*-ary case, the cardinality of $\mathbb{N}^n$ is the same as $\mathbb{N}$.

Hence, the cardinality of $\mathscr{P}(\mathbb{N}^n)$ is the same as the cardinality of $\mathscr{P}(\mathbb{N})$, whatever that might be — recall the independence of the continuum hypothesis.

Taking the union of all of these, i.e.,

$$\bigcup\{\mathscr{P}(\mathbb{N}^n) \mid n \in \mathbb{N}, n > 0\}$$

yields the set of all arithmetic relations.

A countable union of sets of uncountable cardinality κ has cardinality κ.

So, the number of arithmetic relations is the power of the continuum.

Diagonal Arguments

In approaching Gödel's theorems and associated theorems, e.g.,

Tarski's, that the set (of Gödel numbers of sentences in) Th $\mathfrak{N}$ is undefinable in $\mathfrak{N}$, and

Church's, that the set of logically valid sentences is not decidable,

a familiarity with diagonal arguments helps.

(This is "one approach" to these according to Enderton.)

"Diagonalizing out" is best learned by example. So we might as well revisit the first instance historically, viz.. the proof of Cantor's Theorem.

Cantor's Theorem Revisited

Cantor's Theorem. For any set X, $X \prec \mathscr{P}(X)$.

Proof. Obviously $X \preceq \mathscr{P}(X)$ by mapping each element of X into the singleton set containing it. By definition, that $X \prec \mathscr{P}(X)$ means that $X \preceq \mathscr{P}(X)$ but $\mathscr{P}(X) \not\preceq X$. So we need only show that $\mathscr{P}(X) \not\preceq X$. So suppose instead that $\mathscr{P}(X) \preceq X$. Then, by the Schröder-Bernstein Theorem, it follows that $X \sim \mathcal{P}(X)$, and hence there exists a bijection $f : X \to \mathcal{P}(X)$. But we can easily show that there is not even a surjection $f : X \to \mathscr{P}(X)$.

For let $f : X \to \mathscr{P}(X)$. Define $C := \{y \in X \mid y \notin f(y)\}$. If f is onto, then there exists $z \in X$ s.t. $f(z) = C$.

Now ask if $z \in C$. By the definition of C, $z \in C$ iff $z \notin f(z)$, i.e., $z \in C$ iff $z \notin C$. ■

The Characteristic Function of a Set

Fix a set S. For each $A \subseteq S$, i.e., each $A \in \mathscr{P}(S)$, the characteristic function of A is is a mapping $\chi_A : S \to \{0, 1\}$ defined

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

Thus χ_A merely codes for membership in A with 0's and 1's.

CAUTION: Smith uses just the opposite coding. This makes certain arithmetic things down the road slightly simpler, but it goes against the grain of tradition. We will stick with the traditional definition at the expense of minor additional complexity down the road.

For the Case $X = \mathbb{N}$

Suppose $f : \mathbb{N} \to \mathscr{P}(\mathbb{N})$ is onto. Then we can schematically enumerate the characteristic functions of its values as follows:

	0	1	2	$\cdots$	n	$\cdots$
$f(0)$	δ_{00}	δ_{01}	δ_{02}	$\cdots$	δ_{0n}	$\cdots$
$f(1)$	δ_{10}	δ_{11}	δ_{12}	$\cdots$	δ_{1n}	$\cdots$
$f(2)$	δ_{20}	δ_{21}	δ_{22}	$\cdots$	δ_{2n}	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$f(n)$	δ_{n0}	δ_{n1}	δ_{n2}	$\cdots$	δ_{nn}	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

where

$$\delta_{ij} = \begin{cases} 1 & \text{if } j \in f(i) \\ 0 & \text{if } j \notin f(i). \end{cases}$$

For the Case $X = \mathbb{N}$

	0	1	2	$\cdots$	n	$\cdots$
$f(0)$	δ_{00}	δ_{01}	δ_{02}	$\cdots$	δ_{0n}	$\cdots$
$f(1)$	δ_{10}	δ_{11}	δ_{12}	$\cdots$	δ_{1n}	$\cdots$
$f(2)$	δ_{20}	δ_{21}	δ_{22}	$\cdots$	δ_{2n}	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$f(n)$	δ_{n0}	δ_{n1}	δ_{n2}	$\cdots$	δ_{nn}	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Next, let χ_c be the characteristic function of $C := \{n \mid n \notin f(n)\}$ and let N be Boolean negation. Then $\chi_C(n) = N(\delta_{nn})$. Thus, for any $n \in \mathbb{N}$,

$$C = f(n) \text{ iff } N(\delta_{nn}) = \delta_{nn},$$

which is impossible. Thus, $C \notin \text{ran } f$.

Generalization?

Definition. A reductio argument is a diagonal argument iff . . . ? . . .

Enderton's "Self-Reference" Preview

Enderton writes, there are three approaches to Gödel's theorems: (1) self-reference, (2) diagonalization, (3) computability.

Preliminaries for Enderton's "self-reference" approach:

Let $WFF(\mathcal{L}_{PA})$ be the set of wffs of the language $\mathcal{L}_{PA}$ of arithmetic. Let

$$\# : WFF(\mathcal{L}_{PA}) \to \mathbb{N}$$

be a 1-1 function, called a *Gödel numbering* of $WFF(\mathcal{L}_{PA})$, and let $\#\alpha$ be called the Gödel number of the wff α, i.e., the value of α under $\#$. (Actually, $\#$ will be defined from a Gödel numbering of $\mathcal{L}_{PA}$ together with the fixed logical symbols.)

Let $\mathcal{G}$ map finite sequences of wffs into $\mathbb{N}$. Thus, if Σ is a set of sentences of $\mathcal{L}_{PA}$, then every (Hilbert) derivation D from Σ has a Gödel number $\mathcal{G}(D)$. (The range of $\mathcal{G}$ will be disjoint with the range of $\#$.)

Enderton's Self-Reference Preview (cont.)

Theorem. Suppose $A \subseteq \text{Th } \mathfrak{N}$ is such that its associated set of Gödel numbers $\{\#\alpha \mid \alpha \in A\}$ is definable in $\mathfrak{N}$. Then there exists a sentence $\sigma \in \text{Th } \mathfrak{N}$ s.t. $A \nvdash \sigma$ (and thus $\sigma \notin \text{Cn } A$).

In other words, the set of consequences of any arithmetically definable set of arithmetically true sentences is incomplete.

It follows immediately that the set of Gödel numbers of Th $\mathfrak{N}$ is undefinable in $\mathfrak{N}$. (Tarski)

Strategy of the Proof

Strategy.

1. Assume $A \subseteq \text{Th } \mathfrak{N}$ is such that its associated set of Gödel numbers $\{\#\tau \mid \tau \in A\}$ is definable in $\mathfrak{N}$.
2. Since $\text{Cn}(\text{Th } \mathfrak{N}) = \text{Th } \mathfrak{N}$ and Cn is monotonic, $\text{Cn } A \subseteq \text{Th } \mathfrak{N}$.
3. Construct σ so that it "asserts" [is true in $\mathfrak{N}$ iff] $A \nvdash \sigma$ (the tricky part).
4. So, if $A \vdash \sigma$, then σ is false in $\mathfrak{N}$.
5. But then, by the soundness theorem, $\text{Cn } A$ has a member false in $\mathfrak{N}$.
6. This contradicts the assumption that $A \subseteq \text{Th } \mathfrak{N}$.
7. Hence $A \nvdash \sigma$.
8. Since σ asserts $A \nvdash \sigma$, it follows that σ is true in $\mathfrak{N}$, i.e., $\sigma \in \text{Th } \mathfrak{N}$.
9. Therefore, $\sigma \in \text{Th } \mathfrak{N}$ and $A \nvdash \sigma$.

A Relation and a Lemma

Let R_A be the ternary relation on $\mathbb{N}$ s.t.

$\langle a, b, c \rangle \in R_A$ iff there exists a wff α with $FV(\alpha) = \{v_0\}$ and there exists a deduction D from A s.t.
(i) $\#\alpha = a$,
(ii) $\mathcal{G}(D) = c$, and
(iii) the last wff in the sequence D is $\alpha(S^b 0)$, where $\alpha(S^b 0)$ is the result of substituting the numeral $S^b 0$ for the variable v_0 in α.

Lemma. If $\{\#\tau \mid \tau \in A\}$ is definable in $\mathfrak{N}$, then so is R_A.

Pf. To be given at some point.

Proof of the Theorem

Recall just what it is we're trying to prove.

Theorem. Suppose $A \subseteq \text{Th } \mathfrak{N}$ is such that its associated set of Gödel numbers $\{\#\tau \mid \tau \in A\}$ is definable in $\mathfrak{N}$. Then there exists a sentence $\sigma \in \text{Th } \mathfrak{N}$ s.t. $A \nvdash \sigma$ (and thus $\sigma \notin \text{Cn } A$).

Proof.

Proof of the Theorem

(1) Fix A and suppose that $\{\#\tau \mid \tau \in A\}$ is definable in $\mathfrak{N}$.

(2) For any sets Σ, Σ' of sentences of any elementary language, if $\Sigma \subseteq \Sigma'$, then Cn $\Sigma \subseteq$ Cn Σ' (monotonicity of Cn).

Thus Cn $A \subseteq$ Cn Th $\mathfrak{N}$.

But since Th $\mathfrak{N}$ is complete and consistent, Cn Th $\mathfrak{N} =$ Th $\mathfrak{N}$.

Thus, Cn $A \subseteq$ Th $\mathfrak{N}$.

Proof of the Theorem (cont.)

(3) Given that $\{\#\tau \mid \tau \in A\}$ is definable in $\mathfrak{N}$, it follows from the Lemma that R_A is definable in $\mathfrak{N}$. Let $\rho(v_0, v_1, v_2)$ be a formula that defines R_A. Furthermore, let $\theta(v_0)$ be the wff

$$\forall v_2 \neg \rho(v_0, v_0, v_2),$$

and let $q := \#\theta$. Define σ to be the sentence

$$\forall v_2 \neg \rho(S^q 0, S^q 0, v_2).$$

Thus, σ is true in $\mathfrak{N}$ iff there is no number that is the Gödel number of some deduction from A of the result of substituting the numeral $S^q 0$ for v_0 in $\theta(v_0)$, i.e., iff there is no number that is the Gödel number of some deduction of σ from A.

Proof of the Theorem (cont.)

(4) Suppose that, contrary to what σ asserts, that $A \vdash \sigma$. Then there exists a derivation D of σ from A, and hence

$$\models_{\mathfrak{N}} \rho(S^q 0, S^q 0, S^{\mathcal{G}(D)} 0).$$

But obviously,

$$\sigma \models \neg\rho(S^q 0, S^q 0, S^{\mathcal{G}(D)} 0),$$

and so $\not\models_{\mathfrak{N}} \sigma$.

(5) But then, since $A \vdash \sigma$ and hence by soundness $A \models \sigma$, not every member of A can be true in $\mathfrak{N}$.

Proof of the Theorem (cont.)

(6) This contradicts the assumption that $A \subseteq \text{Th}\ \mathfrak{N}$

(7) Therefore, by reductio, $A \nvdash \sigma$.

(8) Hence, for every $k \in \mathbb{N}$, $\langle q, q, k \rangle \notin R_A$, i.e.,

$$\models_{\mathfrak{N}} \neg\rho(S^q 0, S^q 0, S^k 0).$$

Consequently,

$$\models_{\mathfrak{N}} \forall v_2 \neg\rho(S^q 0, S^q 0, v_2),$$

which is just to say that $\models_{\mathfrak{N}} \sigma$.

(9) Therefore, we have $\sigma \in \text{Th}\ \mathfrak{N}$ and $A \nvdash \sigma$, as claimed. ■

Scholium

Scholium. If every decidable set of natural numbers is definable in $\mathfrak{N}$, then it also follows that Th $\mathfrak{N}$ is not axiomatizable.

Characteristic Function of a Set

Def. Suppose we're given some domain D and $A \subseteq D$. E.g., D might be $\mathbb{N}$ or $\mathbb{N} \times \mathbb{N}$. Then the *characteristic function* χ_A of A is the function $\chi_A : D \to \{0, 1\}$ such that for every $x \in D$,

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

This is standard. For some reason, Smith defines the characteristic fn. of a set just the opposite.

Def. *Smith's characteristic function* c_A of A is the function $c_A : D \to \{0, 1\}$ such that for every $x \in D$,

$$c_A(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \notin A. \end{cases}$$

Partial Functions

So far, by a function $f : \mathbb{N} \to \mathbb{N}$ we have meant a function whose domain is all of $\mathbb{N}$. In recursive function theory, it's convenient to talk about partial recursive functions, by which is mean a recursive function that is not defined on all of $\mathbb{N}$.

Def. A **partial numerical function** f is a function s.t. $\operatorname{dom} f \subseteq \mathbb{N}$ and $\operatorname{ran} f \subseteq \mathbb{N}$. A **partial function on** $\mathbb{N}$ is just a function whose domain is a subset of $\mathbb{N}$. If $\operatorname{dom} f = \mathbb{N}$, then f is said to be **total**.

Def. The **numerical domain** of an algorithm Π is just the domain of the partial function f implicitly defined by Π, where $f(n) = a$ iff Π terminates on input n with output a. (In general, $\operatorname{ran} f$ need not be a subset of $\mathbb{N}$.)

A Modest Theorem*

"Theorems" with an asterisk indicate they are not grounded in ZF, but rely on the intuitive concept of an effective procedure.

Theorem*. A set of natural numbers is effectively enumerable iff it is the numerical domain of some algorithm.

Proof*. $\Rightarrow$ Let $n_0, n_1, \ldots$ be an effective enumeration of the set. Rather trivially, let the algorithm be: for each natural number, take its index (i.e., its ordinal place) in the enumeration.

$\Leftarrow$ Let Π be the algorithm. Its domain can be effectively enumerated as follows. Apply one step of the algorithm to 0. If the algorithm terminates, put 0 on the list. Then apply two steps of the algorithm to 0 and to 1, $\ldots$, $n+1$ steps to 0,$\ldots$, n, outputting the input of any terminations.

Obvious Corollaries*

Corollary*. There are (only) countably many effectively enumerable sets.

Proof*. Since our (meta-)language is countable, there are only countably many instructions (finite sequences of symbols) and since any algorithm is a finite set of instructions, only countably many algorithms, and hence, by the theorem, effectively enumerable sets. ■

Corollary*. There are uncountably many sets of natural numbers that are not decidable or even effectively enumerable.

Proof*. This follows by taking note that $\mathcal{P}(\mathbb{N})$ is uncountable.

The Basic Theorem* about Effectively Enumerable Sets

Theorem*. There is an effectively enumerable (e.e.) set whose complement in $\mathbb{N}$ is not effectively enumerable.

Proof*. Let $W_0, W_1, \ldots$ be an enumeration of all e.e. sets. Let $K := \{e \in \mathbb{N} \mid e \in W_e\}$. We show (1) that $\overline{K} := \mathbb{N} \setminus K$ is not e.e., and then (2) that K is.

(1) By the definition of K, $e \in \overline{K}$ iff $e \notin W_e$, for any $e \in \mathbb{N}$. Suppose $\overline{K}$ is e.e. Then $\overline{K} = W_n$ for some n. If $n \in \overline{K}$, then $n \notin W_n$, i.e., $n \notin \overline{K}$. On the other hand, if $n \notin \overline{K}$, then $n \in W_n$, that is $n \in \overline{K}$. So, $\overline{K}$ is not e.e. (Note, this is a diagonal argument.)

Enumeration of the e.e. Sets and Their Contents

	0	1	2	$\cdots$	e	$\cdots$
W_0	δ_{00}	δ_{01}	δ_{02}	$\cdots$	δ_{0e}	$\cdots$
W_1	δ_{10}	δ_{11}	δ_{12}	$\cdots$	δ_{1e}	$\cdots$
W_2	δ_{20}	δ_{21}	δ_{22}	$\cdots$	δ_{2e}	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
W_e	δ_{e0}	δ_{e1}	δ_{e2}	$\cdots$	δ_{ee}	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

where

$$\delta_{ij} = \begin{cases} 1 & \text{if } j \in W_i \\ 0 & \text{if } j \notin W_i \end{cases}$$

Proof that $\overline{K}$ is not e.e.

	0	1	2	$\cdots$	e	$\cdots$
W_0	δ_{00}	δ_{01}	δ_{02}	$\cdots$	δ_{0e}	$\cdots$
W_1	δ_{10}	δ_{11}	δ_{12}	$\cdots$	δ_{1e}	$\cdots$
W_2	δ_{20}	δ_{21}	δ_{22}	$\cdots$	δ_{2e}	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
W_e	δ_{e0}	δ_{e1}	δ_{e2}	$\cdots$	δ_{ee}	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

$K := \{e \in \mathbb{N} \mid e \in W_e\} = \{e \in \mathbb{N} \mid \delta_{ee} = 1\}$, and
$\overline{K} := \{e \in \mathbb{N} \mid e \notin W_e\} = \{e \in \mathbb{N} \mid \delta_{ee} = 0\}$.
Hence, for any $n \in \mathbb{N}$, $n \in \overline{K}$ iff $\delta_{nn} = 0$ iff $n \notin W_n$. Thus $\overline{K} \neq W_n$ for any $n \in \mathbb{N}$.

What the matrix is not!

Don't confuse the previous matrix of 0's and 1's with the following matrix of natural numbers.

	0	1	2	$\cdots$	e	$\cdots$
W_0	$W_0(0)$	$W_0(1)$	$W_0(2)$	$\cdots$	$W_0(e)$	$\cdots$
W_1	$W_1(0)$	$W_1(1)$	$W_1(2)$	$\cdots$	$W_1(e)$	$\cdots$
W_2	$W_2(0)$	$W_2(1)$	$W_2(2)$	$\cdots$	$W_2(e)$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
W_e	$W_e(0)$	$W_e(1)$	$W_e(2)$	$\cdots$	$W_e(e)$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Proof that K is e.e.

(2) That K is effectively enumerable follows from a modification of a familiar procedure:

- Run the algorithm for W_0 for one step, and output 0 should it terminate with 0.
- Run the algorithms for W_0 and W_1, each for two steps, and output 0, resp. 1, should it terminate with 0, resp. 1.

$\vdots$

- In general: run the algorithms for $W_0, \ldots, W_e$, each for $e + 1$ steps, and output 0, ..., e, respectively, should it terminate on 0, ..., e, respectively.

If the procedure spits out e while running the algorithm for W_e, that's only because $e \in W_e$. Conversely, if $e \in W_e$, then after some finite number of steps e will be outputted. ■

Corollary* to the Basic Theorem* about e.e. Sets

Corollary*. There is an e.e. set that is not decidable.

Proof*. Suppose every e.e. set decidable. Then the complement of every e.e. set would be e.e., contradicting the basic theorem.

(An effective procedure for enumerating the complement goes as follows. Go through $\mathbb{N}$ applying the decision procedure. Whenever a number fails the test, put it on the list.)

Sufficiently "Expressive" Languages

Smith gives us the following definition:

> *An interpreted formal language $\mathcal{L}$ is* **sufficiently expressive** *iff (1) it can express every effectively computable one-place numerical function, and (2) it can form wffs that quantify over numbers.*

How do we capture this notion from the point of view of elementary languages? One difficulty is that Smith is not starting off with $\mathcal{L}_{PA}$ and its intended interpretation $\mathfrak{N}$, so condition (2) needs clarification. But so does condition (1), since we have talked about decidable and effectively enumerable sets, but not about computable functions.

At least, by 'an *interpreted* formal language $\mathcal{L}$', we have presumably singled out a structure $\mathfrak{A}$ for $\mathcal{L}$ as the intended interpretation.

Sufficiently "Expressive" Languages (cont.)

Now, as to (2), in order to be able to "quantify over numbers" we must have at a minimum:

(i) $\mathbb{N} \subseteq |\mathfrak{A}|$.

But, more than this, we need a way that $\forall$ quantifies exactly over $\mathbb{N}$. This means that

(ii) there must be a formula $Num(v_0)$ that defines $\mathbb{N}$ in $\mathfrak{A}$.

(Mnemonically, *Num* is to be read "is a natural number".)

Then we can relativize quantification over the natural numbers in the standard way:

(a) $\forall x\ \varphi$ relativized to $\mathbb{N}$ is the wff $\forall x(Num(x) \to \varphi)$.
(b) $\exists x\ \varphi$ relativized to $\mathbb{N}$ is the wff $\exists x(Num(x) \wedge \varphi)$.

This takes care of Smith's condition (2) that the language $\mathcal{L}$ can form wffs that quantify over numbers.

Sufficiently "Expressive" Languages (cont.)

What about Smith's condition (1), viz., that $\mathcal{L}$ can express every effectively computable one-place (total) numerical function?

Note that if $f : \mathbb{N} \to \mathbb{N}$ is computable, then f is a decidable relation. The procedure is: given $\langle n, m \rangle$, compute $f(n)$ and test whether $f(n) = m$.

Conversely, if f is a function and is decidable as a relation, then f is a computable function. The reason is this. Given input n, ask if $\langle n, 0 \rangle \in f$, if $\langle n, 1 \rangle \in f$, and so on. Eventually, for some m, the answer to "is $\langle n, m \rangle \in f$?" will be yes. This is a computation of of the value m for $f(n)$. Thus ...

Sufficiently "Expressive" Languages (cont.)

Lemma. f is a computable numerical function iff f is a single-valued, decidable relation.

Thus condition (1), that every effectively computable one-place numerical function is expressible just amounts to the condition:

(iii) Every decidable binary relation f on $\mathbb{N}$ that is also a function is definable in $\mathfrak{A}$.

(Recall that $\mathfrak{A}$ is the intended model of the interpreted language $\mathcal{L}$.)

Sufficiently "Expressive" Languages (cont.)

To sum up, we have:

"Sufficiently Expressible" Reformulated. Let $\mathcal{L}$ be an elementary language with intended model $\mathfrak{A}$. Then $\mathcal{L}$ is *sufficiently expressible* iff

(i) $\mathbb{N} \subseteq |\mathfrak{A}|$,
(ii) there is a formula $Num(v_0)$ that defines $\mathbb{N}$ in $\mathfrak{A}$, and
(iii) every decidable binary relation R on $\mathbb{N}$ that is also a function is definable in $\mathfrak{A}$.

We now have the following theorem about sufficiently expressible languages.

Smith's 1st Preview Theorem*

Theorem*. The set of truths, i.e., Th $\mathfrak{A}$, of a sufficiently expressible language $\mathcal{L}$ is not e.e.

Proof. By the basic theorem about e.e. sets, there exists an e.e. set of natural numbers whose complement (relative to $\mathbb{N}$) is not e.e. Let K be such an e.e. set.

Since K is e.e., there is a computable function $f : \mathbb{N} \to \mathbb{N}$ that enumerates K.

Hence, since $\mathcal{L}$ is sufficiently expressible, there is then a wff $\kappa(v_0, v_1)$ that defines f in the intended model $\mathfrak{A}$.

Smith's 1st Preview Theorem* (cont.)

Hence, for any $n \in \mathbb{N}$,

$$n \in K \text{ iff } \exists v_0(Num(v_0) \wedge \kappa(v_0, \mathbf{S}^n\mathbf{0})) \in \text{Th } \mathfrak{A},$$

and

$$n \notin K \text{ iff } \neg\exists v_0(Num(v_0) \wedge \kappa(v_0, \mathbf{S}^n\mathbf{0})) \in \text{Th } \mathfrak{A}.$$

For the sake of reductio, now suppose that Th $\mathfrak{A}$ is e.e., and let $\sigma_0, \sigma_1, \ldots$ be an effective enumeration of it.

It then follows that there is an effective procedure for enumerating $\overline{K}$. It goes as follows.

Smith's 1st Preview Theorem* (cont.)

Since for each $n \in \overline{K}$, the sentence

$$\neg \exists v_0 (Num(v_0) \land \kappa(v_0, \mathbf{S}^n\mathbf{0}))$$

is a member of Th $\mathfrak{A}$, that sentence will eventually occur in the effective enumeration $\sigma_0, \sigma_1, \ldots$ of Th $\mathfrak{A}$.

So go through $\sigma_0, \sigma_1, \ldots$, and write down n whenever the sentence

$$\neg \exists v_0 (Num(v_0) \land \kappa(v_0, \mathbf{S}^n\mathbf{0}))$$

appears, for each natural number n.

The result is an effective enumeration of $\overline{K}$, which, by hypothesis, is not e.e. ■

Corollaries to Smith's 1st Preview Theorem

Cor*. The set of sentences true in the intended model $\mathfrak{A}$ of a sufficiently expressive language is not axiomatizable. I.e., Th $\mathfrak{A}$ is not axiomatizable. (From our point of view, Smith's terminology "*effectively* axiomatizable" is pleonastic.)

Proof. Suppose Th $\mathfrak{A}$ were axiomatizable. Then, by the enumerability theorem (for reasonable languages) of elementary logic, it would be e.e., in conflict with the theorem.

Corollary*. Any axiomatizable subtheory of the set of sentences true in the intended model $\mathfrak{A}$, of a sufficiently expressive language is incomplete. (That is, any axiomatizable subtheory of Th $\mathfrak{A}$ is incomplete.)

Proof. Let T be an axiomatizable subtheory of Th $\mathfrak{A}$. Since T is axiomatizable, it is e.e. by the enumerability theorem. So suppose that T is complete. Then $T =$ Th $\mathfrak{A}$. But Th $\mathfrak{A}$ is not e.e.

Representability (Capturability) of a Relation in a Theory

Def. By an **arithmetic language**, we mean a reasonable language (in the sense of Enderton) that has at least an individual constant **0** and a one-place function symbol **S**. (The point is to guarantee that there is a term (numeral) $\mathbf{S}^n\mathbf{0}$ for each $n \in \mathbb{N}$.)

Def. Let T be a theory in an arithmetic language and $R \subseteq \mathbb{N}^{n+1}$. Then R is **representable (capturable)** in T iff there is a wff $\rho(v_0, \ldots, v_n)$ such that for all $k_0, \ldots, k_n \in \mathbb{N}$,

$$\langle k_0, \ldots, k_n \rangle \in R \text{ iff } T \vdash \rho(\mathbf{S}^{k_0}\mathbf{0}, \ldots, \mathbf{S}^{k_n}\mathbf{0})$$

and

$$\langle k_0, \ldots, k_n \rangle \notin R \text{ iff } T \vdash \neg\rho(\mathbf{S}^{k_0}\mathbf{0}, \ldots, \mathbf{S}^{k_n}\mathbf{0}).$$

"Sufficiently Strong" Theories and Smith's 2nd Preview

Def*. A theory T in an arithmetic language is **sufficiently strong** iff every decidable subset of $\mathbb{N}$ is representable in T.

Theorem*. Any consistent, sufficiently strong theory T in an arithmetic language is undecidable.

Proof*. First, effectively enumerate all wffs with exactly v_0 free:

$$\varphi_0(v_0), \varphi_1(v_0), \ldots.$$

Next, define

$$D := \{n \in \mathbb{N} \mid T \vdash \neg\varphi_n(\mathbf{S}^n\mathbf{0})\}.$$

Proof of Smith's 2nd Preview

Now suppose T is decidable.

Then D is a decidable subset of $\mathbb{N}$, and since T is sufficiently strong, D is representable in T.

So, let $\rho(v_0)$ represent D in T. By hypothesis, there exists an n s.t. $\rho(v_0) = \varphi_n(v_0)$.

Now,

$$\begin{aligned} n \in D \quad &\Leftrightarrow \quad T \vdash \neg\varphi_n(\mathbf{S}^n\mathbf{0}) \\ &\Leftrightarrow \quad T \vdash \neg\rho(\mathbf{S}^n\mathbf{0}) \\ &\Leftrightarrow \quad n \notin D. \qquad \blacksquare \end{aligned}$$

Corollary* to Smith's 2nd Preview

Corollary*. Let T be a consistent, sufficiently strong, axiomatizable theory in an arithmetic language. Then T is incomplete.

Proof. By the theorem, T must be undecidable. Now, any complete, axiomatizable theory is decidable. Hence, T must be incomplete.

Concluding Scholium

Scholium. The two previous theorems and their corollaries may seem to close the case for the unaxiomatizability of arithmetic, and thus obviate the need for the rest of the course.

Keep in mind, though, they they are formulated for languages and theories in those languages with "rather strong" properties.

We have not established, for example, that $\mathcal{L}_{PA}$ is "sufficiently expressible," or that there is a theory in $\mathcal{L}_{PA}$ that is "sufficiently strong." Thus, we haven't settled whether Th $\mathfrak{N}$ is axiomatizable.