Fragments of Arithmetic

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What [Theory] is Arithmetic?

Let us stipulate $\mathcal{L}_{PA} = \{\mathbf{0}, \mathbf{S}, \mathbf{+}, \bullet\}$. In principle there are two different sorts of choices:

Semantic determination: Fix the standard model $\mathfrak{N} = (\mathbb{N}, 0, S, +, \cdot)$ and take Th $\mathfrak{N}$.

Axiomatic determination: Fix some set of axioms A and take Cn A.

Historically, the axiomatic method preceded the semantic method. Without having the notion of the semantic method, Gödel showed in 1931 that Cn A is incomplete for a traditional set of axioms due to Dedekind, but commonly associated with the name ‘Peano’, which has subsequently come to be known as Peano Arithmetic. Not only that, but no theory which is an axiomatic extension of Peano Arithmetic is complete (unless inconsistent).

Peano Arithmetic

Axioms of Peano Arithmetic

S1. $\forall x\,(\mathbf{0} \neq \mathbf{S}x)$

S2. $\forall x \forall y(\mathbf{S}x = \mathbf{S}y \rightarrow x = y)$

A1. $\forall x\,(x + \mathbf{0} = x)$

A2. $\forall x \forall y\,(x + \mathbf{S}y = \mathbf{S}(x + y))$

M1. $\forall x\,(x \bullet \mathbf{0} = \mathbf{0})$

M2. $\forall x \forall y\,(x \bullet \mathbf{S}y = x \bullet y + x)$

Induction Schema. For any wff $\varphi(x)$ with at least x free, the universal closure of the following is an axiom:

$$\varphi(0) \rightarrow \forall x(\varphi(x) \rightarrow \varphi(\mathbf{S}x)) \rightarrow \forall x \varphi(x).$$

What Is Arithmetic?

Prior to Gödel 1931, this *was* arithmetic, and still is the basis for (almost all) number theory.

I.e., typical unsolved problems are known to be solvable in Peano Arithmetic.

E.g., Goldbach's conjecture, that any even number greater than 2 is the sum of two primes.

More formally:

$$\forall x(Even(x) \ \wedge \ 2 < x) \rightarrow \exists y \exists z(Prime(y) \wedge Prime(z) \ \wedge \ y + z = x)).$$

Exercise. Define in $\mathfrak{N}$: $Even(\ )$, $<$, $Prime(\ )$.

Fragments of Interest

Before addressing **PA**, we want to address a number of fragments:

1. Robinson's System **Q**, a finitely axiomatizable subtheory of Peano arithmetic.
2. The restriction of a (sub)theory to quantifier free sentences, e.g., Calculator arithmetic, a.k.a., Baby Arithmetic. (Smith)
3. Arithmetic in more restrictive languages. (Enderton)

Why?

- We want to get a sense for what parts of arithmetic do what.
- Robinson's **Q** is the minimal system that displays Gödel type features (representability of recursive functions and thus undecidability)

Robinson's **Q**

Axioms of Robinson's **Q**:

S1. $\forall x\,(\mathbf{0} \neq \mathbf{S}x)$

S2. $\forall x \forall y(\mathbf{S}x = \mathbf{S}y \rightarrow x = y)$

S3. $\forall x(x \neq \mathbf{0} \rightarrow \exists y(x = \mathbf{S}y))$

A1. $\forall x\,(x + \mathbf{0} = x)$

A2. $\forall x \forall y\,(x + \mathbf{S}y = \mathbf{S}(x + y))$

M1. $\forall x\,(x \bullet \mathbf{0} = \mathbf{0})$

M2. $\forall x \forall y\,(x \bullet \mathbf{S}y = x \bullet y + x)$

Relation to **PA**

Heuristic. Robinson's **Q** $\approx$ Peano Arithmetic w/o the induction schema.

Lemma. Robinson's **Q** is a subtheory of **PA**.

Proof. *Exercise*.

The "oddity" of Robinson's **Q**:

- Robinson's **Q** is remarkably strong for computational purposes (representability of decidable relations and computable functions), yet
- remarkably weak from a number-theoretic perspective (e.g., it does not entail that addition is commutative).

Calculator Arithmetic (a.k.a. Baby Arithmetic)

Characterization I: The set of quantifier-free sentences of Th $\mathfrak{N}$.

Characterization II: Work in a logic with no variables. Use the following axiom schemata, CA: For $n, m \in \mathbb{N}$:

Schema S1 $\mathbf{0} \neq \mathbf{S}^{n+1}\mathbf{0}$

Schema S2 $\mathbf{S}^{n+1}\mathbf{0} = \mathbf{S}^{m+1}\mathbf{0} \to \mathbf{S}^{n}\mathbf{0} = \mathbf{S}^{m}\mathbf{0}$

Schema A1 $\mathbf{S}^{n}\mathbf{0} \mathbf{+} \mathbf{0} = \mathbf{S}^{n}\mathbf{0}$

Schema A2 $\mathbf{S}^{n}\mathbf{0} \mathbf{+} \mathbf{S}^{m+1}\mathbf{0} = \mathbf{S}(\mathbf{S}^{n}\mathbf{0} \mathbf{+} \mathbf{S}^{m}\mathbf{0})$

Schema M1 $\mathbf{S}^{n}\mathbf{0} \bullet \mathbf{0} = \mathbf{0}$

Schema M2 $\mathbf{S}^{n}\mathbf{0} \bullet \mathbf{S}^{m+1}\mathbf{0} = (\mathbf{S}^{n}\mathbf{0} \bullet \mathbf{S}^{m}\mathbf{0}) \mathbf{+} \mathbf{S}^{n}\mathbf{0}$

Calculator Arithmetic (cont.)

Theorem. The characterizations are equivalent, i.e., the set of quantifier-free consequences of the axiom schemata just is the set of quantifier-free sentences of Th $\mathfrak{N}$..

The proof will be given as a sequence of lemmata.

Lemma 1. For all $m, n \in \mathbb{N}$, if $m \neq n$, then $\mathrm{CA} \vdash \mathbf{S}^m\mathbf{0} \neq \mathbf{S}^n\mathbf{0}$.

Proof. Suppose, without loss of generality, that $m < n$. We proceed by induction on m (in the metalanguage).

Case $m = 0$. Then $0 \neq \mathbf{S}^n\mathbf{0}$ is an instance of S1 (since there exists $k \in \mathbb{N}$ s.t. $n = k + 1$).

Calculator Arithmetic: Lemma 1 (cont.)

Proof (cont.).

Inductive Step: Pick m arbitrarily. The inductive hypothesis is that for every $n > m$, $\mathrm{CA} \vdash \mathbf{S}^m\mathbf{0} \neq \mathbf{S}^n\mathbf{0}$. We want to show this holds for $m+1$. So, suppose $m+1 < k$. Then, since $k \neq 0$, $k = n+1$ for some n. Since $m+1 < k = n+1$, we have $m < n$. Hence, by the inductive hypothesis, $\mathrm{CA} \vdash \mathbf{S}^m\mathbf{0} \neq \mathbf{S}^n\mathbf{0}$. From this, by modus tollens on S2, viz.,

$$\mathbf{S}^{m+1}\mathbf{0} = \mathbf{S}^{n+1}\mathbf{0} \rightarrow \mathbf{S}^m\mathbf{0} = \mathbf{S}^n\mathbf{0},$$

we get $\mathrm{CA} \vdash \mathbf{S}^{m+1}\mathbf{0} \neq \mathbf{S}^{n+1}\mathbf{0}$, i.e., $\mathrm{CA} \vdash \mathbf{S}^{m+1}\mathbf{0} \neq \mathbf{S}^k\mathbf{0}$. ■

Calculator Arithmetic: Lemmata 2 and 3

Change of notation: following Smith, we write $\overline{n}$ for the numeral $\mathbf{S}^n\mathbf{0}$.

Lemma 2. For all $m, n \in \mathbb{N}$, CA $\vdash \overline{m}\mathbf{+}\overline{n} = \overline{m + n}$.

Proof. By induction (in the metalanguage) on n. Details are an exercise. *Suggestion*: First take m to be an arbitrary natural number. Then do the induction on the parameter n.

Lemma 3. For all $m, n \in \mathbb{N}$, CA $\vdash \overline{m} \bullet \overline{n} = \overline{m \cdot n}$.

Proof. *Exercise*.

Calculator Arithmetic: Lemmata 4

Lemma 4. For any variable free term τ, CA $\vdash \tau = \overline{n}$, where n is the denotation of τ in $\mathfrak{N}$ (i.e., where $\overline{s}(\tau) = n$ for any mapping s of variables into $|\mathfrak{N}|$).

Proof. *Exercise*. *Suggeston*: Do induction on the shape of τ.

Calculator Arithmetic: Characterization Theorem

Theorem. For any quantifier free sentence σ,

$$\mathsf{CA} \vdash \sigma \Leftrightarrow \sigma \in \mathsf{Th}\ \mathfrak{N}$$

and

$$\mathsf{CA} \vdash \neg\sigma \Leftrightarrow \sigma \notin \mathsf{Th}\ \mathfrak{N}.$$

Proof. *Exercise*.

Calculator Arithmetic: Corollaries

Corollary. For any quantifier-free sentence σ, either $\mathrm{PA} \vdash \sigma$ or $\mathrm{PA} \vdash \neg\sigma$.

Proof. *Exercise*.

Corollary. The set of quantifier free arithmetic truths is decidable.

Proof. *Exercise*.

Pure Successor Arithmetic

Let $\mathcal{L}_S := \{\mathbf{0}, \mathbf{S}\}$ and let $\mathfrak{N}_S = (\mathbb{N}, 0, S)$ be the reduct of $\mathfrak{N}$ to $\mathcal{L}_S$. What can we say about Th $\mathfrak{N}_S$?

Consider the following (infinite) set A_S of axioms:

S1. $\forall x\, (\mathbf{0} \neq \mathbf{S}x)$
S2. $\forall x \forall y (\mathbf{S}x = \mathbf{S}y \rightarrow x = y)$
S3. $\forall x (x \neq \mathbf{0} \rightarrow \exists y (x = \mathbf{S}y))$
S4.1 $\forall x\, (\mathbf{S}x \neq x)$
S4.2 $\forall x\, (\mathbf{SS}x \neq x)$

$\vdots$

S4.n $\forall x\, (\mathbf{S}^n x \neq x)$

$\vdots$

Models of A_S

Clearly Cn $A_S \subseteq$ Th $\mathfrak{N}_S$.

What can we say about arbitrary models $\mathfrak{A} = (|\mathfrak{A}|, \mathbf{0}^{\mathfrak{A}}, \mathbf{S}^{\mathfrak{A}})$ of Cn A_S?

First, for any model $\mathfrak{A}$ of A_S there is an isomorphic embedding $\phi : \mathbb{N} \to |\mathfrak{A}|$ of $\mathfrak{N}_S$ into $\mathfrak{A}$ such that

$$\phi(n) = (\mathbf{S}^{\mathfrak{A}})^n(\mathbf{0}^{\mathfrak{A}}).$$

Proof. *Exercise*. Suggestion: Build up ϕ in stages, beginning with $\phi_0 = \{\langle 0, \mathbf{0}^{\mathfrak{A}}\rangle\}$. For each $n \in \mathbb{N}$, let $\phi_{n+1} = \phi_n; \langle n+1, (\mathbf{S}^{\mathfrak{A}})^{n+1}\mathbf{0}^{\mathfrak{A}}\rangle$. Then show that $\phi = \bigcup_{n\in\mathbb{N}} \phi_n$ has the requisite properties to be the isomorphic embedding in question.

Models of A_S (cont.)

Second, the embedding need not be onto, i.e., $|\mathfrak{A}|$ can contain non-standard elements (elements not named by any numeral).

Proof. *Exercise*. Suggestion: Add an individual constant c to $\mathcal{L}_S$ and join to A_S the infinite set

$$\Sigma := \{\mathbf{S}^n\mathbf{0} \neq c \mid n \in \mathbb{N}\}$$

of sentences of the expanded language. Then apply the compactness theorem.

Models of A_S (cont.)

Third, if a model $\mathfrak{A}$ of A_S has a non-standard element a, it has at least a countable infinity of non-standard elements, viz., for each n an element $a_n = (\mathbf{S}^{\mathfrak{A}})^n(a)$ as well as an element b_n s.t. $a = (\mathbf{S}^{\mathfrak{A}})^n(b_n)$.

Pictorially, the action of $\mathbf{S}^{\mathfrak{A}}$ is as follows:

$$\cdots \mapsto b_2 \mapsto b_1 \mapsto a \mapsto a_1 \mapsto a_2 \mapsto \cdots .$$

Since there can be no loops (by the schema S4), this is called a Z-chain, since it is isomorphic to the integers $\mathbb{Z}$ under successor on the integers.

Models of A_S (cont.)

Fourth. A model of A_S can have any number of Z-chains.

Proof. Add any number of new constants to $\mathcal{L}_S$. Let C be the set of new constants. For each $c \in C$, add to A_S all sentences of the form $S^n 0 \neq c$, and for distinct constants $c, c' \in C$ sentences of the form $\mathbf{S}^n c \neq c'$ as well as $\mathbf{S}^n c' \neq c$. Compactness tells us there is a model whose reduct to $\mathcal{L}_S$ contains a Z-chain for each new constant.

Fifth. Any structure $\mathfrak{B}$ for $\mathcal{L}_{\mathcal{S}}$ that consists of an isomorphic copy of $\mathfrak{N}_S$ together with any number of Z-chains is a model of A_S. (Check that each of the axioms is true in $\mathfrak{B}$.)

Models of A_S (cont.)

Lemma. Any two Z-chains are isomorphic.

Proof. *Exercise*.

Theorem (AC). Any two models of A_S with the same number of Z-chains are isomorphic.

Pf. Choose a 1-1 correspondence between Z-chains. Choose for each pairing of Z-chains an isomorphism between these. Take the union over these mappings together with an isomorphism between the standard parts.

Models of A_S (cont.)

Corollary (AC). Any two uncountable models of A_S with the same cardinality are isomorphic.

Proof. If the models have cardinality $\lambda > \aleph_0$, then they each have λ many Z-chains.

Theorem (AC). Cn A_S is a complete theory.

Pf. According to the Łoś-Vaught test, if a theory with no finite models is λ-categorical for some infinite cardinal λ, then it is complete. Cn A_S has no finite models and is λ-categorical in any uncountable λ.

Models of A_S (cont.)

Corollary (AC). Cn $A_S = \text{Th}\ \mathfrak{N}_S$.

Corollary (AC). Cn A_S is decidable.

Short of effectively enumerating the language and Cn A_S by brute force, is there some reasonably elegant decision procedure?

Also, can we get rid of the dependence of these results on AC?

Elimination of Quantifiers

Defn. A theory T **admits elimination of quantifiers** iff for every wff φ there is a quantifier-free wff ψ such that

$$T \models (\varphi \leftrightarrow \psi).$$

Keep in mind that this includes wffs with free variables. But if T admits elimination of quantifiers, every *sentence* has a quantifier free equivalent under T.

Elimination of Quantifiers (cont.)

Lemma. Suppose that for every wff φ of the form

$$\exists x(\alpha_0 \wedge \cdots \wedge \alpha_n),$$

where $\alpha_0, \ldots, \alpha_n$ are either atomic or the negation of atomic wffs, there is a quantifier-free wff ψ such that

$$T \models (\varphi \leftrightarrow \psi).$$

Then T admits elimination of quantifiers.

Elimination of Quantifiers (cont.)

Proof. We show the more limited result that, granting the supposition, there is a quantifier-free equivalent for every wff of the form $\exists x \theta$ where θ is quantifier free.

Proof of the general case is left as an exercise. (Suggestion: Recall that every wff has an equivalent in prenex normal form. Use induction.)

We can put θ into disjunctive normal form so that $\exists x \theta$ is equivalent to a wff of the form

$$\exists x((\beta_1 \wedge \cdots \wedge \beta_m) \vee (\gamma_1 \wedge \cdots \wedge \gamma_\ell) \vee \cdots \vee (\eta_1 \wedge \cdots \wedge \eta_k)).$$

Elimination of Quantifiers (cont.)

Since existential quantification distributes over disjunction, this is equivalent to

$$\exists x(\beta_1 \wedge \cdots \wedge \beta_m) \vee \exists x(\gamma_1 \wedge \cdots \wedge \gamma_\ell) \vee \cdots \vee \exists x(\eta_1 \wedge \cdots \wedge \eta_k).$$

By assumption, each disjunct is equivalent to a quantifier-free wff, so the disjunction is equivalent to a quantifier-free wff. ■

Scholium

Lemma: In the case of a *complete* (and consistent) theory T, $T = \text{Th}\ \mathfrak{A}$ for any $\mathfrak{A} \in \text{Mod}\ T$. Pick any such $\mathfrak{A}$. In this special case, T admits elimination of quantifiers iff for any wff φ there is a quantifier-free wff ψ s.t.

$$\models_{\mathfrak{A}} (\varphi \leftrightarrow \psi)\ \ [s]$$

for any mapping s of variables into the domain of discourse of $\mathfrak{A}$.

Proof: Exercise. [*Hint*: Try induction on the shape of φ.]

Elimination of Quantifiers in Th $\mathfrak{N}_S$

Theorem. Th $\mathfrak{N}_S$ admits elimination of quantifiers.

Proof. Since Th $\mathfrak{N}_S$ is complete, it suffices to show that any wff of the form

$$\exists x(\alpha_0 \wedge \cdots \wedge \alpha_n),$$

where $\alpha_0, \ldots, \alpha_n$ are either atomic or the negation of atomic wffs, there is a quantifier-free wff ψ s.t.

$$\models_{\mathfrak{N}_S} (\exists x(\alpha_0 \wedge \cdots \wedge \alpha_n) \leftrightarrow \psi) \;\; [s],$$

for any mapping s of variables into $\mathbb{N}$.

Proof of Elimination of Quantifiers in Th $\mathfrak{N}_S$ (cont.)

(1) We can assume x occurs in each α_i, since if x does not occur in α then $\exists x(\alpha \wedge \beta)$ is equivalent to $(\alpha \wedge \exists x \beta)$, and thus we only need to find an equivalent for $\exists x \beta$.

(2) Thus each α_i has one of the following four forms:

$$\mathbf{S}^n x = \mathbf{S}^m 0,\ \mathbf{S}^n x \neq \mathbf{S}^m 0,\ \mathbf{S}^n x = \mathbf{S}^m y,\ \mathbf{S}^n x \neq \mathbf{S}^m y,$$

wherein we can assume that $y \neq x$, since in that case we could replace $\mathbf{S}^n x = \mathbf{S}^m x$ with $\mathbf{0} = \mathbf{0}$ if $n = m$ and $\mathbf{0} \neq \mathbf{0}$ otherwise, and vice-versa for $\mathbf{S}^n x \neq \mathbf{S}^m x$.

Proof of Elimination of Quantifiers in Th $\mathfrak{N}_S$ (cont.)

(3) There are two cases now to consider.

Case (i): Each α_i is an inequality. Then the entire wff can be replaced by $\mathbf{0} = \mathbf{0}$, since no matter what the values of n, m and y are, x can be chosen sufficiently large.

E.g., suppose we have

$$\exists x((\mathbf{S}^7 x \neq \mathbf{S}^9\mathbf{0}) \wedge (x \neq \mathbf{S}y) \wedge (\mathbf{S}^9 x \neq \mathbf{S}^3 z)).$$

Pick any assignment s of values to y and z, say $y = 5$ and $z = 7$. Then $s(x) = 6$ is big enough in order that

$$\models_{\mathfrak{N}_S} \exists x((\mathbf{S}^7 x \neq \mathbf{S}^9\mathbf{0}) \wedge (x \neq \mathbf{S}y) \wedge (\mathbf{S}^9 x \neq \mathbf{S}^3 z)) \ [s].$$

Proof of Elimination of Quantifiers in Th $\mathfrak{N}_S$ (cont.)

Case (ii). At least one α_i is an equation, say $\mathbf{S}^m x = t$, where t does not contain x.

Since the solution for x must be non-negative, replace this by

$$t \neq \mathbf{0} \ \wedge \cdots \wedge t \neq \mathbf{S}^{m-1}\mathbf{0},$$

if $m > 0$ or else by $\mathbf{0} = \mathbf{0}$ if $m = 0$.

Then in each other α_j replace $\mathbf{S}^k x = u$ (resp., $\mathbf{S}^k x \neq u$) first with $\mathbf{S}^{k+m} x = \mathbf{S}^m u$ (resp., $\mathbf{S}^{k+m} x \neq \mathbf{S}^m u$) and then (since $\mathbf{S}^m x = t$) with $\mathbf{S}^k t = \mathbf{S}^m u$ (resp, $\mathbf{S}^k t \neq \mathbf{S}^m u$). Then x no longer appears in any of the α's, so the quantifier can be dropped. ■

Example of Case (ii) in the Proof

Example. Take

$$\exists x(S^m x = y \wedge S^k x = z)$$

The procedure is to form first

$$(y \neq 0 \wedge \cdots \wedge y \neq S^{m-1}0) \wedge S^{k+m} x = S^m z.$$

Then, since $S^{k+m} x = S^k S^m x = S^k y$, we can eliminate x:

$$(y \neq 0 \wedge \cdots \wedge y \neq S^{m-1}0) \wedge S^k y = S^m z.$$

This has the same solutions for y and z simultaneously as the initial existential wff.

Elimination of Quantifiers in Cn A_S Without Assuming Completeness

Problem. Using the same procedure for finding for a given φ a quantifier free ψ, show that

$$A_S \models (\varphi \leftrightarrow \psi)$$

without assuming the completeness of Cn A_S.

Solution: *Exercise*. *Suggestion*: Show that for any wff φ there is a quantifier free ψ s.t. for any assignment s of numbers to variables:

$$\models_{\mathfrak{N}} (\varphi \leftrightarrow \psi) \quad [s].$$

Alternative proof of completeness of Cn A_S w/o AC

Lemma. Cn A_S is complete (without assuming AC).

Proof. Any atomic sentence is an equation of the form $\mathbf{S}^m\mathbf{0} = \mathbf{S}^n 0$ and either it or it's negation is entailed by A_S depending on whether or not $m = n$.

Thus, any quantifier-free sentence is such that either it or it's negation is entailed by A_S.

Now let σ be any sentence of the language. By elimination of quantifiers, there is a quantifier-free τ, s.t. $A_S \models (\sigma \leftrightarrow \tau)$.

If $A_S \models \tau$, then $A_S \models \sigma$, and, if $A_S \models \neg\tau$, then $A_S \models \neg\sigma$. Thus, $A_S \models \sigma$ or $A_S \models \neg\sigma$. ■

Implications of Elimination of Quantifiers for Definability

Lemma. Any relation definable in $\mathfrak{N}_S$ is definable by a quantifier free formula.

Proof. This follows immediately from the elimination of quantifiers.

Corollary. Only finite and cofinite subsets of $\mathbb{N}$ are definible in $\mathfrak{N}_S$. (A set is cofinite iff it's relative complement is finite.)

Proof. *Exercise*.

Corollary. The relation $\{\langle m, n\rangle \in \mathbb{N} \times \mathbb{N} \mid m < n\}$ is not definable in $\mathfrak{N}_S$.

Proof. *Exercise*.

Ordered Successor Arithmetic

Let $\mathcal{L}_L := \mathcal{L}_S ; <$.

Let $\mathfrak{N}_L$ be the expansion of $\mathfrak{N}_S$ to a structure for $\mathcal{L}_L$ such that $<$ is interpreted as the less-than relation on $\mathbb{N}$. I.e., let $\mathfrak{N}_L = (\mathbb{N}, 0, S, <)$.

Now attempt to axiomatize $\mathrm{Th}\, \mathfrak{N}_L$.

Notational abbreviations: $x \not< y$ for $\neg x < y$, and $x \leq y$ for $(x < y \ \vee \ x = y)$.

An Axiom System A_L in the Language $\mathcal{L}_L$

Consider the following finite set A_L of Axioms.

L5 $\forall x \forall y \forall z (x < y \to y < z \to x < z)$

L4 $\forall x \forall y (x < y \to y \not< x)$

L3 $\forall x \forall y (x < y \ \vee \ y < x \ \vee \ x = y)$

L2 $\forall x \ (x \not< 0)$

L1 $\forall x \forall y (x < Sy \leftrightarrow x \leq y)$

S3 $\forall x (x \neq 0 \to \exists y (x = Sy))$

Some Theorems of A_L

What about the rest of the axioms we had in play for A_S? Let's consider some fairly immediate theorems of A_L:

1. $A_L \vdash \forall x(x < Sx)$
2. $A_L \vdash \forall x\ (x \not< x)$
3. $A_L \vdash \forall x \forall y(x \not< y \leftrightarrow y < x \ \vee \ x = y)$
4. $A_L \vdash \forall x \forall y(x < y \leftrightarrow Sx < Sy)$
5. $A_L \vdash$ S1, where S1 is $\forall x\ (\mathbf{0} \neq \mathbf{S}x)$
6. $A_L \vdash$ S2, where S2 is $\forall x \forall y(\mathbf{S}x = \mathbf{S}y \rightarrow x = y)$
7. $A_L \vdash$ S4.n for all $n \in \mathbb{N}$ s.t. $n > 0$, where S4.n is $\forall x\ \mathbf{S}^n x \neq x$

Proofs: *Exercises*. Suggestion: Use natural deduction. Enderton (p. 194) gives some hints as to which axioms to use in each case.

Models of A_L

A word about the models of A_L ...

- Since Cn A_L contains Cn A_S as a subtheory, the reduct $\mathfrak{A} \restriction \mathcal{L}_S$ of any model $\mathfrak{A}$ of A_L to the the language $\mathcal{L}_S$ is a model of A_S.
- Hence, $\mathfrak{A} \restriction \mathcal{L}_S$ consists of a standard part plus 0 or more Z-chains.
- Each Z-chain has the standard order on the integers.
- But, for any (strict linear) order type that has the standard part first, there is a model $\mathfrak{A}$ of A_L with that order type between Z-chains.
- In particular, this means that Cn A_L is λ-categorical in *no* infinite cardinal λ. Thus, there is no hope of applying the Łoś-Vaught test in order to prove completeness. But ...

Elimination of Quantifiers for Cn A_L

Theorem. Cn A_L admits elimination of quantifiers.

Proof. Again, it suffices to show that for any wff of the form $\exists x(\beta_0 \wedge \cdots \wedge \beta_k)$, where each β_i is atomic or the negation of an atomic wff, there is a quantifier-free wff ψ s.t.

$$A_L \models (\exists x(\beta_0 \wedge \cdots \wedge \beta_n) \leftrightarrow \psi).$$

The atomic wff's are all of the form $S^m u = S^n t$ or $S^m u < S^n t$, where u and t may be variables or 0.

We proceed in steps:

Proof of Elimination of Quantifiers for Cn A_L (cont.)

(1) All negated wffs can be replaced by equivalent disjunctions of atomic wffs.

- replace $S^m u \neq S^n t$ with $S^m u < S^n t \vee S^n t < S^m u$
- replace $S^m u \not< S^n t$ with $S^m u = S^n t \vee S^n t < S^m u$

The result is a wff $\exists x \theta$ that is no longer in the form $\exists x(\beta_0 \wedge \cdots \wedge \beta_k)$. But θ can be put into disjunctive normal form $(\delta_1 \vee \cdots \vee \delta_\ell)$, where each δ_i is a conjunction of atomic wffs. But since,

$$\exists x(\delta_1 \vee \cdots \vee \delta_\ell) \vdash \dashv (\exists x \delta_1 \vee \cdots \vee \exists x \delta_\ell),$$

the problem reduces to finding a quantifier free equivalent for wffs of the form

$$\exists x(\alpha_1 \wedge \cdots \wedge \alpha_p),$$

where each α_i is atomic.

Proof of Elimination of Quantifiers for Cn A_L (cont.)

(2) As in the case of quantifier elimination for Successor Arithmetic, we can assume x occurs in each α_i, since if x does not occur in α, then

$$\exists x(\alpha \land \beta) \vdash\dashv \alpha \land \exists x \beta.$$

Next, for any α_i in which x occurs twice:

- replace $S^m x = S^n x$ with $0 = 0$ if $m = n$ else $0 \neq 0$, and
- replace $S^m x < S^n x$ with $0 = 0$ if $m < n$ else $0 \neq 0$.

Thus, any remaining α_i in which x occurs can be taken to be of the form $S^m x = S^n t$, $S^m x < S^n t$ or $S^n t < S^m x$, where t is either 0 or some variable other than x.

Proof of Elimination of Quantifiers for Cn A_L (cont.)

(3) If some remaining α_i is an equation, proceed as in the elimination of quantifiers for Successor Arithmetic. I.e.,

- replace the equation $S^m x = S^n t$ (where t is 0 or a variable) with $S^n t \neq 0 \wedge \cdots \wedge S^n t \neq S^{m-1}0$ ($m > 0$, else $0 = 0$).
- replace any other equation $S^k x = S^\ell u$ with $S^{k+m} x = S^{\ell+m} u$, any inequality $S^k x < S^\ell u$ with $S^{k+m} x < S^{\ell+m} u$, and any inequality $S^\ell u < S^k x$ with $S^{\ell+m} u < S^{k+m} x$.
- since $S^m x = S^n t$, replace these in turn with $S^{k+n} t = S^{\ell+m} u$, $S^{k+n} t < S^{\ell+m} u$, or $S^{\ell+m} u < S^{k+n} t$, respectively.

The variable x no longer appears in any term and the result ψ of deleting the quantifier is such that

$$A_L \models \exists x(\alpha_0 \wedge \cdots \wedge \alpha_p) \leftrightarrow \psi.$$

Proof of Elimination of Quantifiers for Cn A_L (cont.)

(4) If each α_i is an inequality of the form $S^n t < S^m x$, then there is no upper bound on x and thus always a solution q for the conjunction of the inequalities. And, the claim is (*exercise*), this is a logical consequence of A_L, i.e.,

$$A_L \models \exists x(\alpha_1 \wedge \cdots \wedge \alpha_p).$$

Consequently,

$$A_L \models (\exists x(\alpha_1 \wedge \cdots \wedge \alpha_p) \leftrightarrow \psi),$$

for any logical consequence ψ of A_L. (Take $0 = 0$ if you like.)

Proof of Elimination of Quantifiers for Cn A_L (cont.)

(5) If each α_i is an inequality of the form $S^m x < S^n t$, then there need be no positive lower bound on x, and we can simply substitute 0 for x in each term. Then

$$A_L \models \exists x(\alpha_1(x) \wedge \cdots \wedge \alpha_p(x)) \leftrightarrow (\alpha_1(0) \wedge \cdots \wedge \alpha_p(0))$$

Either side of the biconditional may be true or false in $\mathfrak{N}_L$, or any other model $\mathfrak{A}$, but it is a consequence of A_L that their truth covaries in a model.

Proof of Elimination of Quantifiers for Cn A_L (cont.)

(6) Otherwise we have mixed inequalities, for example

$$(t_1 < S^m x) \wedge (S^n x < t_2),$$

where t_1 and t_2 in general are complex terms. Replace this first with

$$(S^n t_1 < S^{n+m} x) \wedge (S^{m+n} x < S^m t_2).$$

Since $S^{n+m}x = S^{m+n}x$, we expect $S^{n+1}t_1 < S^m t_2$. But also, from $S^n x < t_2$, we expect $S^n 0 < t_2$. So, the full replacement for the inequality is

$$(S^{n+1} t_1 < S^m t_2) \wedge (S^n 0 < t_2).$$

If there are more than two terms we sum over all the "exponents" in the intermediate step. ■

Completeness of Cn A_L

Cor. Cn A_L is complete.

Pf. First, argue that for any atomic sentence σ, Either $A_L \models \sigma$ or $A_L \models \neg\sigma$. In other words for all $m, n \in \mathbb{N}$ either $A_L \models S^m 0 = S^n 0$ or $A_L \models S^m 0 \neq S^n 0$, and $A_L \models S^m 0 < S^n 0$ or $A_L \models S^m 0 \not< S^n 0$. Next, argue that the same holds for any quantifier-free sentence. Finally, let τ be an arbitrary sentence of the language. By the elimination of quantifiers, there exists a quantifier-free sentence χ s.t.

$$A_L \models (\tau \leftrightarrow \chi).$$

Since either $A_L \models \chi$ or $A_L \models \neg\chi$, it follows that either $A_L \models \tau$ or $A_L \models \neg\tau$. ■

Definability Results for $\mathfrak{N}_L$

Cor. Only finite and cofinite subsets of $\mathbb{N}$ are definable in $\mathfrak{N}_L$.

Pf. *Exercise*. Variant on an exercise about Successor Arithmetic. Uses the consequence of quantifier-eliminability that any definable relation is definable by a quantifier-free wff. ■

Cor. Addition is not definable in $\mathfrak{N}_L$.

Pf. Suppose it is. Then the set of even numbers is definable by $\{n \in \mathbb{N} \mid \exists y(y + y = n)\}$. (More rigorously, suppose the wff $\varphi(v_0, v_1, v_2)$ defines the addition relation. Then $\exists y \varphi(y, y, v_0)$ defines the evens.) But the set of evens is neither finite nor cofinite, in conflict with the preceding corollary. ■

Final Corollary for Cn A_L

Cor*. Cn A_L is decidable.

Pf. Any complete axiomatizable theory in a reasonable language is decidable.

Adding Addition

Let

$$\mathcal{L}_{Add} = \mathcal{L}_L; +,$$

and let

$$\mathfrak{N}_A = (\mathbb{N}, 0, S, <, +).$$

Can we set out axioms for Th $\mathfrak{N}_A$?

Instead ask directly if Th $\mathfrak{N}_A$ is decidable. (Just as with any complete theory in a reasonable language, Th $\mathfrak{N}_A$ is axiomatizable iff decidable.)

But first, what do the models of Th $\mathfrak{N}_A$ look like?

Models of Th $\mathfrak{N}_A$

Suppose there is at least one Z-chain.

Claim: *There must be infinitely many larger Z-chains*.

Let a be an element of a Z chain. Clearly $a < a + a$. But since, any $a' \geq a$ in the same Z-chain is s.t. $a + n = a'$ for some $n \in \mathbb{N}$, $a + a$ lies in a different Z-chain greater than the first.

Claim: *There must be infinitely many smaller Z-chains*.

$\forall x(\exists y((y + y = x) \vee (y + y = Sx))$ is true in $\mathfrak{N}_A$. (Every number is either "even" or "odd"). So for any non-standard a there is a b in a smaller Z-chain such that $b + b = a$ or $b + b = S(a)$.

Models of Th $\mathfrak{N}_A$ (cont.)

Claim: *Between any two Z-chains there must be another*.

Let a_1 and a_2 be from distinct Z-chains, with $a_1 < a_2$. Now, $a_1 + a_2$ is either "even" or "odd." I.e., there exists a non-standard b s.t. $b + b = a_1 + a_2$ or $b + b = S(a_1 + a_2)$. In either case b must belong to a distinct Z-chain s.t. $a_1 < b < a_2$.

Scholium. The order of Z-chains might be that of the rationals or of the reals, but could also be something more exotic, e.g., a series of Z-chains ordered as the rationals followed by a series of Z-chains ordered as the reals. In general, Th $\mathfrak{N}_A$ is not λ-categorical in any uncountable λ.

The Decidability of Th $\mathfrak{N}_A$

Theorem (Presburger 1929). Th $\mathfrak{N}_A$ is decidable.

Proof Strategy.

1. Add to the language for each $n \geq 2$ the symbol $\equiv_n$ for equivalence mod n.
2. Show that Th $\mathfrak{N}^{\equiv}$, where

$$\mathfrak{N}^{\equiv} = (\mathbb{N}, 0, S, <, +, \equiv_2, \equiv_3, \ldots),$$

   admits elimination of quantifiers.
3. Show that for any sentence σ one can effectively find a quantifier-free sentence τ s.t. Th $\mathfrak{N}_A \models \sigma \leftrightarrow \tau$.
4. Show that the truth (in $\mathfrak{N}^{\equiv}$) of any quantifier-free sentence is decidable.

This yields an effective procedure for deciding the truth (in $\mathfrak{N}_A$) of any sentence in the language $\mathcal{L}_A$. ■

Multiplication Is Not Definable in $\mathfrak{N}_A$

Defn. A set D of natural numbers is *periodic* iff there exists $p \in \mathbb{N}$ s.t. for every $n \in \mathbb{N}$, $n \in D$ iff $n + p \in D$.

Defn. D is *eventually periodic* there exist $p, M \in \mathbb{N}$ s.t. for every $n > M$, $n \in D$ iff $n + p \in D$.

Theorem. A set of numbers is definable in $\mathfrak{N}_A$ iff it is eventually periodic.

Cor. Multiplication is not definable in $\mathfrak{N}_A$.

Pf. If if it were, the set of squares would be definable in $\mathfrak{N}_A$, but the set of squares is not eventually periodic.