A Brief Diversion

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Where We're Headed

Keep in mind our current objective.

- We saw that Q is incomplete.
- We continued studying Q because we claimed Q would turn out to be "sufficiently strong."
- Short of a full theory of decidability, we'll introduce a limited version of computable functions, viz. the primitive recursive functions.
- Will show that any primitive recursive function is representable in Q.
- The same will hold for Peano Arithmetic since it extends Q.
- This will be enough to show that Peano Arithmetic is incomplete.

But First a Diversion

Cat and Mouse Game: How might we "minimally" extend Q to get a complete theory?

- The idea is to add an induction scheme, but as weak as we can get away with.
- First try: $I\Delta_0$, i.e., allow only Δ_0 wffs in the induction scheme.
- Result: Get a lot of encouraging theorems, e.g., commutativity and associativity of addition, $\leq$ is a total ordering of $\mathbb{N}$, etc.
- But $I\Delta_0$ remains "trivially" incomplete. We can find a wff $\varepsilon(x, y)$ defining the relation $\{\langle m, n\rangle \mid 2^m = n\}$. (This is not obvious. It follows from two later results: (i) $f(m) = 2^m$ is primitive recursive (p.r.), and (ii) any p.r. function is definable in $\mathfrak{N}$.) However $Q \nvdash \forall x \exists y \varepsilon(x, y)$.

Diversion (cont.)

- Next try: $I\Sigma_1$, i.e., to allow Σ_1 wffs in the induction scheme.
- Game fizzles at this point, since Smith doesn't show this is a case of "trivial" incompleteness.
- Move on immediately to full Peano Arithmetic (PA). Ask
  - Doesn't it seem it should be complete?
  - Doesn't it seem it is consistent?