

The Representability of the P.R. Functions in Q

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Weak Representability of a Function in a Theory

Recall that an $(n + 1)$ -ary numeric *relation* R is represented in an arithmetic¹ theory T by the wff $\rho(v_0, \dots, v_n)$ iff for all $m_0, \dots, m_n \in \mathbb{N}$,

1. $\langle m_0, \dots, m_n \rangle \in R$ iff $T \vdash \rho(\overline{m_0}, \dots, \overline{m_n})$, and
2. $\langle m_0, \dots, m_n \rangle \notin R$ iff $T \vdash \neg \rho(\overline{m_0}, \dots, \overline{m_n})$,

where $\overline{m_i}$ is the numeral for m_i (short for $S^{m_i}0$). This suggests the following notion for the representation of a **function** in a theory.

Def. A function $f : \mathbb{N}^n \rightarrow \mathbb{N}$, is **weakly represented** by the wff $\varphi(v_0, \dots, v_n)$ in T just in case $\varphi(v_0, \dots, v_n)$ represents f in T as an $(n + 1)$ -ary relation.

¹That a theory is arithmetic means only that it has a numeral for each natural number.

(Full) Representability of a Function in a Theory

For notational simplicity, let us restrict ourselves to functions of one variable. The generalization to several variables is straightforward.

It's conceivable that $\varphi(v_0, v_1)$ weakly represents f in T yet T doesn't explicitly entail that each number has a unique value under f , i.e., there may be an m such that

$$T \not\vdash \exists! y \varphi(\bar{m}, y)$$

This motivates the following.

Defn. $\varphi(v_0, v_1)$ **(fully) represents** $f : \mathbb{N} \rightarrow \mathbb{N}$ in T iff for all $m, n \in \mathbb{N}$,

1. if $f(m) = n$, then $T \vdash \varphi(\bar{m}, \bar{n})$, and
2. $T \vdash \exists! y \varphi(\bar{m}, y)$.

A “Dovetailing” Result

Theorem. A property is representable in a theory T just in case its characteristic function is (fully) representable in T .

Proof. \Rightarrow . Let $\varphi(v_0)$ represent P in T . Let $\psi(v_0, v_1)$ be the wff

$$(\varphi(v_0) \wedge v_1 = 1) \vee (\neg\varphi(v_0) \wedge v_1 = 0).$$

The claim is that ψ represents χ_P in T . For suppose that $\chi_P(n) = 1$. Then $n \in P$ and, hence, $T \vdash \varphi(\bar{n})$. Trivially $T \vdash 1 = 1$, so

$$T \vdash (\varphi(\bar{n}) \wedge 1 = 1) \vee (\neg\varphi(\bar{n}) \wedge 1 = 0).$$

That is, $T \vdash \psi(\bar{n}, 1)$, as required.

A “Dovetailing” Result (cont.)

It remains to be shown that for any n , $T \vdash \exists! y \psi(\bar{n}, y)$. Suppose $n \in P$. Then $T \vdash \psi(\bar{n}, 1)$, and so $T \vdash \exists y \psi(\bar{n}, y)$. Now, reasoning in the object language, suppose $\psi(\bar{n}, c)$ for some individual c , i.e.,

$$(\varphi(\bar{n}) \wedge c = 1) \vee (\neg\varphi(\bar{n}) \wedge c = 0).$$

Since $T \vdash \varphi(\bar{n})$, we can infer the l.h. disjunct, and hence that $c = 1$. Hence if $n \in P$, then $T \vdash \exists! y \psi(\bar{n}, y)$. A similar line of reasoning applies in the case in which $n \notin P$.

\Leftarrow . Suppose that $\theta(v_0, v_1)$ represents χ_P in T . Let $\gamma(v_0)$ be the wff

$$\exists y (y = 1 \wedge \theta(v_0, y)).$$

We claim that $\gamma(v_0)$ represents P in T .

A “Dovetailing” Result (cont.)

If $n \in P$, then $T \vdash \theta(\bar{n}, 1)$, and thus

$$T \vdash \exists y(y = 1 \wedge \theta(\bar{n}, y)),$$

i.e., $T \vdash \gamma(\bar{n})$.

Suppose now $n \notin P$. Then $T \vdash \theta(\bar{n}, 0)$. But since θ fully represents χ_P , we have also that $T \vdash \exists! y \theta(\bar{n}, y)$. The supposition that

$$\exists y(y = 1 \wedge \theta(\bar{n}, y))$$

entails $\theta(\bar{n}, 1)$, leading to a contradiction. Hence,

$$T \vdash \neg \exists y(y = 1 \wedge \theta(\bar{n}, y)),$$

i.e., $T \vdash \neg \gamma(\bar{n})$. ■

(Full) Representability in T Entails Weak Representability in T if T Entails Distinct Numbers are Distinct

Lemma. Suppose that for distinct $m, n \in \mathbb{N}$, $T \vdash \bar{m} \neq \bar{n}$. Then, if φ (fully) represents f in T , then φ weakly represents f in T .

Proof. Assume that φ (fully) represents f in T . We need to show that for all $m, n \in \mathbb{N}$

1. if $f(m) = n$, then $T \vdash \varphi(\bar{m}, \bar{n})$, and
2. if $f(m) \neq n$, then $T \vdash \neg\varphi(\bar{m}, \bar{n})$.

The first of these is given immediately. The second is established as follows. Suppose $f(m) \neq n$. From the hypothesis that φ (fully) represents f in T , we have immediately that (i) $T \vdash \varphi(\bar{m}, \overline{f(m)})$ and (ii) $T \vdash \exists! y \varphi(\bar{m}, y)$. Since $f(m) \neq n$, $T \vdash \overline{f(m)} \neq \bar{n}$. On pain of contradiction, it follows that $T \vdash \neg\varphi(\bar{m}, \bar{n})$. ■

An Equivalent Condition: Functional Representability

Def. Say that $\varphi(v_0, v_1)$ **functionally represents** f in T iff for all $m, n \in \mathbb{N}$, if $f(m) = n$, then $T \vdash \forall y(\varphi(\bar{m}, y) \leftrightarrow y = \bar{n})$.

Lemma. Suppose T is at least as strong as Calculator Arithmetic (Baby Arithmetic). Then $\varphi(v_0, v_1)$ (fully) represents f in T iff $\varphi(v_0, v_1)$ functionally represents f in T .

Pf. EXERCISE

Weak Representability in Q Entails Representability in Q

Lemma. Suppose T is at least as strong as Q. Then, if f is weakly representable in T , it is (fully) representable in T .

Proof. Suppose that φ weakly represents f in T . Let $\tilde{\varphi}$ be the wff

$$\varphi(v_0, v_1) \wedge (\forall z \leq v_1)(\varphi(v_0, z) \rightarrow z = v_1).$$

The claim is that $\tilde{\varphi}$ represents f in T . We need to prove

1. if $f(m) = n$, then $T \vdash \tilde{\varphi}(\bar{m}, \bar{n})$, and
2. for any m , $T \vdash \exists! y \tilde{\varphi}(\bar{m}, y)$

For (1), assume $f(m) = n$. By weak representability we're given that $T \vdash \varphi(\bar{m}, \bar{n})$.

Weak Representability in Q Entails Representability (cont.)

By weak representability we're also given for each $k < n$, $T \vdash \neg\varphi(\bar{m}, \bar{k})$, and thus $T \vdash (\varphi(\bar{m}, \bar{k}) \rightarrow \bar{k} = \bar{n})$. And trivially $T \vdash (\varphi(\bar{m}, \bar{n}) \rightarrow \bar{n} = \bar{n})$. Hence, by order adequacy of Q (clause (4)), we have that

$$T \vdash (\forall z \leq \bar{n})(\varphi(\bar{m}, z) \rightarrow z = \bar{n}).$$

Thus, $T \vdash \tilde{\varphi}(\bar{m}, \bar{n})$.

For (2), i.e., $T \vdash \exists!y\tilde{\varphi}(\bar{m}, y)$ for each m , pick m arbitrarily. By existential generalization, we'll be done if we can show

$$T \vdash \tilde{\varphi}(\bar{m}, \bar{n}) \wedge \forall u(\tilde{\varphi}(\bar{m}, u) \rightarrow u = \bar{n}).$$

The first conjunct has already been established, so we need only show that

$$T \vdash \forall u(\tilde{\varphi}(\bar{m}, u) \rightarrow u = \bar{n}).$$

Weak Representability in Q Entails Representability (cont.)

Reasoning within the object language, pick c to be an arbitrary individual, and suppose that $\tilde{\varphi}(\bar{m}, c)$, i.e., that

$$\varphi(\bar{m}, c) \wedge (\forall z \leq c)(\varphi(\bar{m}, z) \rightarrow z = c).$$

Now, by order adequacy of Q (clause (8)), we have either (i) $c \leq \bar{n}$ or (ii) $\bar{n} \leq c$.

Case (i): We've already shown that

$$T \vdash (\forall z \leq \bar{n})(\varphi(\bar{m}, z) \rightarrow z = \bar{n}).$$

So in particular, $(\varphi(\bar{m}, c) \rightarrow c = \bar{n})$. Since we have $\varphi(\bar{m}, c)$ above by supposition, it follows that $c = \bar{n}$.

Weak Representability in Q Entails Representability (cont.)

Case (ii): By supposition

$$(\forall z \leq c)(\varphi(\bar{m}, z) \rightarrow z = c).$$

We have now both $\bar{n} \leq c$ and $\varphi(\bar{m}, \bar{n})$. Hence it follows that $\bar{n} = c$.

So, in either case, having supposed $\tilde{\varphi}(\bar{m}, c)$, we get that $\bar{n} = c$. Thus, $(\tilde{\varphi}(\bar{m}, c) \rightarrow \bar{n} = c)$. Since c was chosen arbitrarily, we conclude that $\forall u(\tilde{\varphi}(\bar{m}, u) \rightarrow \bar{n} = u)$. So

$$T \vdash \tilde{\varphi}(\bar{m}, \bar{n}) \wedge \forall u(\tilde{\varphi}(\bar{m}, u) \rightarrow \bar{n} = u).$$

By existential generalization,

$$T \vdash \exists y(\tilde{\varphi}(\bar{m}, y) \wedge \forall u(\tilde{\varphi}(\bar{m}, u) \rightarrow y = u)).$$

I.e., $T \vdash \exists! y \tilde{\varphi}(\bar{m}, y)$. ■

Strong Representability of Functions

There is an apparently stronger notion of representability than (full) representability, viz.,

Def. $\varphi(v_0, v_1)$ **strongly represents** $f : \mathbb{N} \rightarrow \mathbb{N}$ in T iff for all $m, n \in \mathbb{N}$,

1. if $f(m) = n$, then $T \vdash \varphi(\bar{m}, \bar{n})$, and
2. $T \vdash \forall x \exists! y \varphi(x, y)$.

The motivation for strong representability is that if φ *strongly* represents f , then we can take φ to *define* f and add a new symbol \mathbf{f} with the axiom

$$\forall x \forall y (\mathbf{f}x = y \leftrightarrow \varphi(x, y))$$

without creating any new consequences in the original language (*conservative extension*).

(Full) Representability Entails Strong Representability

Lemma. If f is (fully) representable in T , then f is strongly representable in T .

Proof. Suppose $\varphi(v_0, v_1)$ (fully) represents f in T . Then,

$$\vec{\varphi}(v_0, v_1) =_{df} (\varphi(v_0, v_1) \wedge \exists!z\varphi(v_0, z)) \vee (v_1 = 0 \wedge \neg\exists!z\varphi(v_0, z))$$

strongly represents f in T . This we must show.

(1) Suppose $f(m) = n$. Then, since φ represents f in T , $T \vdash \varphi(\bar{m}, \bar{n})$ and $T \vdash \exists!z\varphi(\bar{m}, z)$. Hence $T \vdash \vec{\varphi}(\bar{m}, \bar{n})$, which is the first requirement for strong representation.

Representability Entails Strong Representability (cont.)

(2) For the second, we need to show that $T \vdash \forall x \exists! y \vec{\varphi}(x, y)$. So let a be an arbitrary individual. We're done if we can show that $\exists! y \vec{\varphi}(a, y)$. We argue by separation of cases. Either (i) $\exists! y \varphi(a, y)$ or (ii) $\neg \exists! y \varphi(a, y)$.

Case (i): Suppose $\exists! y \varphi(a, y)$. Then for some individual b , $\varphi(a, b)$. That, together with the supposition, entails $\vec{\varphi}(a, b)$. Now suppose $\vec{\varphi}(a, b')$, i.e.,

$$(\varphi(a, b') \wedge \exists! y \varphi(a, y)) \vee (y = 0 \wedge \neg \exists! y \varphi(a, y)).$$

Since we're under the supposition that $\exists! y \varphi(a, y)$, we can eliminate the r.h. disjunct, leaving us with $\varphi(a, b') \wedge \exists! y \varphi(a, y)$. Hence, $\varphi(a, b')$ and in turn, $b' = b$. Therefore $\exists! y \vec{\varphi}(a, y)$.

Representability Entails Strong Representability (cont.)

Case (ii): Suppose $\neg\exists!y\varphi(a, y)$. Note that $\vec{\varphi}(a, 0)$ is the wff:

$$(\varphi(a, 0) \wedge \exists!y\varphi(a, y)) \vee (0 = 0 \wedge \neg\exists!y\varphi(a, y)).$$

Since $0 = 0$, we have the r.h. disjunct, and hence the entire disjunction $\vec{\varphi}(a, 0)$. Now suppose that for some b , $\vec{\varphi}(a, b)$, i.e.,

$$(\varphi(a, b) \wedge \exists!y\varphi(a, b)) \vee (b = 0 \wedge \neg\exists!y\varphi(a, b)).$$

Since we're under the supposition that $\neg\exists!y\varphi(a, y)$, we can infer the r.h. disjunct, and thus $b = 0$. So, again, $\exists!y\vec{\varphi}(a, y)$. ■

A Last Result

The “Last” Lemma. Suppose that T is at least as strong as Q . Let $\varphi(v_0, v_1)$ be any wff with v_0 and v_1 free, and let $\tilde{\varphi}(v_0, v_1)$ be defined as above, i.e.,

$$\tilde{\varphi}(v_0, v_1) =_{df} \varphi(v_0, v_1) \wedge (\forall z \leq v_1)(\varphi(v_0, z) \rightarrow z = v_1).$$

Then for any $n \in \mathbb{N}$,

$$T \vdash \forall x \forall y (\tilde{\varphi}(x, \bar{n}) \wedge \tilde{\varphi}(x, y) \rightarrow y = \bar{n}).$$

Pf. Suppose for arbitrary a and b that $\tilde{\varphi}(a, \bar{n})$ and $\tilde{\varphi}(a, b)$. Since Q is order adequate, $T \vdash b \leq \bar{n} \vee \bar{n} \leq b$. Argue by separation of cases inside T as follows.

A Last Result (cont.)

(1) Suppose $b \leq \bar{n}$. From $\tilde{\varphi}(a, \bar{n})$ we have

$$(\forall z \leq \bar{n})(\varphi(a, z) \rightarrow z = \bar{n}).$$

And from $\tilde{\varphi}(a, b)$, we have $\varphi(a, b)$. Thus, it follows that $b = \bar{n}$.

(2) Suppose $\bar{n} \leq b$. Now from $\tilde{\varphi}(a, b)$, we have

$$(\forall z \leq b)(\varphi(a, z) \rightarrow z = b).$$

And from $\tilde{\varphi}(a, \bar{n})$, we have $\varphi(a, \bar{n})$. Thus, it follows that $\bar{n} = b$. ■

P.R. Adequacy

Def. A theory T is **p.r. adequate** just in case any p.r. function is representable in T .

N.B. If T is p.r. adequate, then any p.r. property or relation is also representable in T . (Recall that a relation R is p.r. iff χ_R is p.r., and χ_R is (fully) representable in T iff R is representable in T .)

Theorem. Q is p.r. adequate.

Proof Strategy. We show that

1. 0, S , and the projection functions are representable in Q,
2. the composition of any functions representable in Q is representable in Q, and finally
3. if g and h are representable in Q and f is defined from g and h by primitive recursion, then f is representable in Q.

Q Is P.R. Adequate

Step 1: EXERCISE

Step 2: We illustrate for the case of functions of a single variable.

Suppose that $\gamma(v_0, v_1)$ represents g in Q and $\eta(v_0, v_1)$ represents h in Q .

Let $\varphi(v_0, v_1)$ be the wff

$$\exists z(\gamma(v_0, z) \wedge \eta(z, v_1)).$$

Then, we claim, $\varphi(v_0, v_1)$ represents $f = h \circ g$ in Q .

To prove this, we need to establish that if $f(m) = n$, then

(i) $Q \vdash \varphi(\bar{m}, \bar{n})$, and (ii) $Q \vdash \exists! y \varphi(\bar{m}, y)$.

So, suppose that $f(m) = n$.

(i) We need to show that $Q \vdash \varphi(\bar{m}, \bar{n})$, or, unpacking φ , that

$$Q \vdash \exists z(\gamma(\bar{m}, z) \wedge \eta(z, \bar{n})).$$

Let $k = g(m)$. By supposition, we have both that $Q \vdash \gamma(\bar{m}, \bar{k})$ and that $Q \vdash \eta(\bar{k}, \bar{n})$. So,

$$Q \vdash (\gamma(\bar{m}, \bar{k}) \wedge \eta(\bar{k}, \bar{n})).$$

Hence, by existential generalization,

$$Q \vdash \exists z(\gamma(\bar{m}, z) \wedge \eta(z, \bar{n})),$$

as required.

(ii) We need to show $Q \vdash \exists! y \varphi(\bar{m}, y)$, i.e., that

$$Q \vdash \exists y(\varphi(\bar{m}, y) \wedge \forall u(\varphi(\bar{m}, u) \rightarrow u = y)).$$

Q Is P.R. Adequate, Step 2 (cont.)

Making φ explicit, we must show that

$$Q \vdash \exists y(\exists z(\gamma(\bar{m}, z) \wedge \eta(z, y)) \wedge \forall u(\exists z(\gamma(\bar{m}, z) \wedge \eta(z, u)) \rightarrow u = y)).$$

For this, it suffices to show

$$Q \vdash \exists z(\gamma(\bar{m}, z) \wedge \eta(z, \bar{n})) \wedge \forall u(\exists z(\gamma(\bar{m}, z) \wedge \eta(z, u)) \rightarrow u = \bar{n}),$$

since the line above it then follows immediately by existential generalization on the constant \bar{n} . That Q entails the first conjunct has already been shown. As for the second conjunct, we reason inside Q. Suppose $\exists z(\gamma(\bar{m}, z) \wedge \eta(z, u))$ for arbitrary u . By existential instantiation we have $\gamma(\bar{m}, b) \wedge \eta(b, u)$ for some b . Since $Q \vdash \exists! y \gamma(\bar{m}, y)$ and $Q \vdash \gamma(\bar{m}, \bar{k})$, we have that $b = \bar{k}$, and since $Q \vdash \exists! y \eta(\bar{k}, y)$ and $Q \vdash \eta(\bar{k}, \bar{n})$, we obtain that $u = \bar{n}$. ■

Q Is P.R. Adequate, Step 3

Step 3: Again, we illustrate with a simple case. Let

$$\begin{aligned}f(0) &= g \\ f(S(x)) &= h(f(x)),\end{aligned}$$

where $g \in \mathbb{N}$. Again, let $\eta(v_0, v_1)$ represent h in Q. And again, let's go through the "translation" procedure to see how to arrive at a wff that defines f in \mathfrak{N} . We have that $\langle n, m \rangle \in f$ iff there exists a sequence of numbers k_0, \dots, k_n s.t.

- ▶ $k_0 = g$,
- ▶ for all $j < n$, $k_{S(j)} = h(k_j)$, and
- ▶ $k_n = m$.

Translating this into β -function talk, we have:

Q Is P.R. Adequate, Step 3 (cont.)

$\langle n, m \rangle \in f$ iff there exist numbers c, d s.t.

- ▶ $\beta(c, d, 0) = g$
- ▶ for all $j < n$, $\beta(c, d, S(j)) = h(\beta(c, d, j))$, and
- ▶ $\beta(c, d, n) = m$.

Let $B(v_0, v_1, v_2, v_3)$ define β in \mathfrak{N} . Then $\langle n, m \rangle \in f$ iff the wff $F(\bar{n}, \bar{m})$ defined as:

$\exists x \exists y \{$

- ▶ $B(x, y, 0, \bar{g})$
- ▶ $(\forall z < \bar{n}) \exists u \exists w [B(x, y, Sz, u) \wedge B(x, y, z, w) \wedge \eta(w, u)]$
- ▶ $B(x, y, \bar{n}, \bar{m})\}$.

is true in \mathfrak{N} .

Q Is P.R. Adequate, Step 3 (cont.)

Recall that $\beta(c, d, i) = rm(c, d(i + 1) + 1)$, so that we can take $B(v_0, v_1, v_2, v_3)$ to be the wff:

$$(\exists x \leq v_0)((v_0 = x \cdot S(v_1 \cdot Sv_2) + v_3) \wedge (v_3 \leq v_1 \cdot Sv_2)).$$

Some facts:

- ▶ $B(v_0, v_1, v_2, v_3)$ is a Δ_0 wff.
- ▶ Thus, since Q correctly decides all Δ_0 wffs, $B(v_0, v_1, v_2, v_3)$ *weakly* represents β in Q.
- ▶ Since $B(v_0, v_1, v_2, v_3)$ is a Δ_0 wff, $F(\bar{n}, \bar{m})$ is a Σ_1 wff.
- ▶ Since Q is Σ_1 complete, $Q \vdash F(\bar{n}, \bar{m})$.

Unfortunately, $\neg F(\bar{n}, \bar{m})$ is **not** Σ_1 , so we cannot conclude [immediately] that $Q \vdash \neg F(\bar{n}, \bar{m})$, and thus that F weakly represents f .

Q Is P.R. Adequate, Step 3 (cont.)

However, we do have that $\tilde{B}(v_0, v_1, v_2, v_3)$, i.e.,

$$B(v_0, v_1, v_2, v_3) \wedge (\forall y \leq v_3)(B(v_0, v_1, v_2, y) \rightarrow y = v_3),$$

(fully) represents β in Q. But more importantly, we have as an instance of the “Last” Lemma:

$$Q \vdash \forall x \forall y \forall z \forall u (\tilde{B}(x, y, z, \bar{m}) \wedge \tilde{B}(x, y, z, u) \rightarrow u = \bar{m}).$$

If we use \tilde{B} rather than B in the definition of $F(v_0, v_1)$, then we can establish not only that

- ▶ $Q \vdash F(\bar{n}, \bar{m})$, assuming $f(n) = m$,

but also that for any n

- ▶ $Q \vdash \exists! y F(\bar{n}, y)$,

and hence that F (fully) represents f in Q.

Q Is P.R. Adequate, Step 3 (cont.)

Thus we redefine $F(\bar{n}, \bar{m})$ to be:

$\exists x \exists y \{$

- ▶ $\tilde{B}(x, y, 0, \bar{g})$
- ▶ $(\forall z < \bar{n}) \exists u \exists w [\tilde{B}(x, y, Sz, u) \wedge \tilde{B}(x, y, z, w) \wedge \eta(w, u)]$
- ▶ $\tilde{B}(x, y, \bar{n}, \bar{m})\}$.

Since this again is a Σ_1 wff (and Q is Σ_1 complete), we have immediately that if $f(n) = m$, then $Q \vdash F(\bar{n}, \bar{m})$.

Thus, what remains to be shown is that for any n , $Q \vdash \exists! y F(\bar{n}, y)$, i.e.,

$$Q \vdash \exists y (F(\bar{n}, y) \wedge \forall v (F(\bar{n}, v) \rightarrow v = y)).$$

Since we already have that $Q \vdash F(\bar{n}, \bar{m})$, the existence part follows immediately, and all we have to do is to show that

$$Q \vdash \forall v (F(\bar{n}, v) \rightarrow v = \bar{m}).$$

Q Is P.R. Adequate, Step 3 (cont.)

This we do by (weak) induction on n .

Base Case ($n = 0$): Here we want to establish

$$Q \vdash \forall v (F(0, v) \rightarrow v = \bar{m}),$$

where it so happens that $m = g$. Arguing now inside of Q, let u be an arbitrary individual and assume that $F(0, u)$. We want to show that $u = \bar{g}$. Now, $F(0, u)$ is a grand existential claim of the form $\exists x \exists y \varphi(x, y, 0, u)$. So, by way of existential instantiation, let c and d be s.t. $\varphi(c, d, 0, u)$. The wff $\varphi(c, d, 0, u)$ is the conjunction of three wffs, the first and last of which are:

- ▶ $\tilde{B}(c, d, 0, \bar{g})$ and
- ▶ $\tilde{B}(c, d, 0, u)$

From these and the “Last” Lemma, it follows that $u = \bar{g}$.

Q Is P.R. Adequate, Step 3 (cont.)

Inductive Step: Now suppose that if $f(n) = m$, then

$$Q \vdash \forall v (F(\bar{n}, v) \rightarrow v = \bar{m}).$$

We want to show that if $f(n) = m$, i.e., if $f(n+1) = h(m)$, then

$$Q \vdash \forall v (F(\overline{n+1}, v) \rightarrow v = \overline{h(m)}).$$

Arguing inside Q, let v be an arbitrary individual s.t. $F(\overline{n+1}, v)$, i.e., s.t. $\exists x \exists y \varphi(x, y, \overline{n+1}, v)$. Let c and d be such an x and y (existential instantiation). That gives us:

- ▶ (1) $\tilde{B}(c, d, 0, \bar{g})$
- ▶ (2) $(\forall z < \overline{n+1}) \exists u \exists w [\tilde{B}(c, d, Sz, u) \wedge \tilde{B}(c, d, z, w) \wedge \eta(w, u)]$
- ▶ (3) $\tilde{B}(c, d, \overline{n+1}, v)$.

Q Is P.R. Adequate, Step 3 (cont.)

Note that (2) entails

$$\blacktriangleright (2') (\forall z < \bar{n}) \exists u \exists w [\tilde{B}(c, d, Sz, u) \wedge \tilde{B}(c, d, z, w) \wedge \eta(w, u)]$$

Keep this tucked away for later use. Next, universally instantiate (2) with \bar{n} and then existentially instantiate for u and w with p and q , leaving us with the three conjuncts:

- $\blacktriangleright (4) \tilde{B}(c, d, S\bar{n}, p),$
- $\blacktriangleright (5) \tilde{B}(c, d, \bar{n}, q),$
- $\blacktriangleright (6) \eta(q, p).$

Conjoin (1), (2'), and (5), and existentially generalize on c and d . That gives us:

Q Is P.R. Adequate, Step 3 (cont.)

$\exists x \exists y \{$

- ▶ $\tilde{B}(x, y, 0, \bar{g})$
- ▶ $(\forall z < \bar{n}) \exists u \exists w [\tilde{B}(x, y, Sz, u) \wedge \tilde{B}(x, y, z, w) \wedge \eta(w, u)]$
- ▶ $\tilde{B}(x, y, \bar{n}, q)\}$.

This is nothing other than $F(\bar{n}, q)$. So, since by the inductive hypothesis

$$Q \vdash \forall x (F(\bar{n}, x) \rightarrow x = \bar{m}),$$

we have $q = \bar{m}$. Thus, from (5) and (6) we get

- ▶ (5') $\tilde{B}(c, d, \bar{n}, \bar{m})$ and
- ▶ (6') $\eta(\bar{m}, p)$

Q Is P.R. Adequate, Step 3 (cont.)

By hypothesis, η represents h in Q, which means that

$$Q \vdash \eta(\overline{m}, \overline{h(m)})$$

and

$$Q \vdash \exists! y \eta(\overline{m}, y).$$

Thus, from (6') we get that $p = \overline{h(m)}$. Plugging this into (4) gives us

$$\blacktriangleright (4') \tilde{B}(c, d, S\overline{n}, \overline{h(m)}).$$

This together with

$$\blacktriangleright (3) \tilde{B}(c, d, \overline{n+1}, v)$$

yields by the “Last” Lemma that $v = \overline{h(m)}$ as desired. ■

Catalogue

Def. $\varphi(v_0, v_1)$ **weakly represents** $f : \mathbb{N} \rightarrow \mathbb{N}$, in T iff $\varphi(v_0, v_1)$ represents f in T as a relation.

Def. $\varphi(v_0, v_1)$ **(fully) represents** $f : \mathbb{N} \rightarrow \mathbb{N}$ in T iff for all $m, n \in \mathbb{N}$, if $f(m) = n$, then

1. $T \vdash \varphi(\bar{m}, \bar{n})$, and
2. $T \vdash \exists! y \varphi(\bar{m}, y)$.

Def. $\varphi(v_0, v_1)$ **functionally represents** f in T iff for all $m, n \in \mathbb{N}$, if $f(m) = n$, then $T \vdash \forall y (\varphi(\bar{m}, y) \leftrightarrow y = \bar{n})$.

Def. $\varphi(v_0, v_1)$ **strongly represents** $f : \mathbb{N} \rightarrow \mathbb{N}$ in T iff for all $m, n \in \mathbb{N}$,

1. if $f(m) = n$, then $T \vdash \varphi(\bar{m}, \bar{n})$, and
2. $T \vdash \forall x \exists! y \varphi(x, y)$.

Representability Interrelations