

The Arithmetization of Syntax

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Assigning Gödel Numbers to Symbols

Let g be a 1-1 mapping from the symbols of \mathcal{L}_{PA} plus logical symbols into \mathbb{N} . Just about any assignment will do, e.g.:

)	(\neg	\rightarrow	\forall	=	0	S	+	\times
1	3	5	7	9	11	13	15	17	19

with $g(v_n) = 2n$ for individual variables.

To every expression $\varepsilon = s_0 s_1 \cdots s_n$ (sequence of symbols) we will assign a Gödel number

$$\#\varepsilon = \pi_0^{g(s_0)+1} \pi_1^{g(s_1)+1} \cdots \pi_n^{g(s_n)+1},$$

where π_i is the i -th prime number, and to every sequence $d = \langle \varphi_0, \varphi_1, \dots, \varphi_n \rangle$ of expressions a super-Gödel number

$$\mathcal{G}(d) = \pi_0^{\#\varphi_0+1} \pi_1^{\#\varphi_1+1} \cdots \pi_n^{\#\varphi_n+1}.$$

Arithmetizing the Syntax

By “arithmetizing the syntax” we mean not just assigning Gödel numbers, but also defining the following arithmetic properties and relations and showing that they are representable in \mathcal{Q} and *a fortiori* in PA.

- ▶ $\{\#t \mid t \text{ is a term}\}$
- ▶ $\{\#\varphi \mid \varphi \text{ is a wff}\}$
- ▶ $\{\langle g(x), \#\varphi \rangle \mid x \text{ is a variable free in } \varphi\}$
- ▶ $\{\#\sigma \mid \sigma \text{ is a sentence}\}$
- ▶ $\{\langle g(x), \#t, \#\varphi, \#\psi \rangle \mid \psi = \varphi_t^x\}$
- ▶ $\{\langle \#\varphi, \#\psi \rangle \mid \psi \text{ is a universal generalization of } \varphi\}$
- ▶ $\{\#\varphi \mid \varphi \text{ is an axiom}\}$
- ▶ $\{\langle \#\varphi, \#\psi, \#\chi \rangle \mid \langle \varphi, \psi, \chi \rangle \in MP\}$
- ▶ $\{\langle \mathcal{G}(d), \#\varphi \rangle \mid d \text{ is a derivation of } \varphi \text{ in PA}\}$

Corner Quotes

Notation. Let $\# \alpha$ be the Gödel number of the expression α . Then $\ulcorner \alpha \urcorner$ is the *numeral* for $\# \alpha$ in \mathcal{L}_{PA} .¹

So, e.g., if the wff $Term(v_0)$ represents the set of Gödel numbers of terms of \mathcal{L}_{PA} , then the sentence $Term(\ulcorner \mathbf{S0} \urcorner)$ says (indirectly) that **S0** is a term.

¹Smith uses this notation indiscriminately for Gödel numbers and their numerals.

Diagonalization

Gödel's construction of a sentence that (indirectly) asserts its own unprovability relies on a diagonal construction. The most direct form of this would be to substitute $\ulcorner \alpha(v_0) \urcorner$ for v_0 in $\alpha(v_0)$ itself to obtain the sentence $\alpha(\ulcorner \alpha(v_0) \urcorner)$. For technical reasons the *diagonalization* of $\alpha(v_0)$ is defined to be

$$\exists v_0 (v_0 = \ulcorner \alpha(v_0) \urcorner \wedge \alpha(v_0)).$$

Thus, there's a function *diag* that takes us from Gödel nos. of wffs to their diagonalizations. We want to also show that that's p.r. and hence representable in \mathcal{Q} .

Fundamentals

We want a function $code : \mathbb{N}^{<\omega} \rightarrow \mathbb{N}$ that is 1-1 and thus assigns a “code” $\#$ to each finite sequence of natural numbers:

$$code(\langle k_0, \dots, k_n \rangle) = \prod_{i=0}^n \pi_i^{k_i+1},$$

with the courtesy value $code(\langle \rangle) = 1$ for the empty sequence.

(Note: this is an easy proof that the set of all finite sequences on a countable set is countable.) So, our assignment of Gödel numbers to expressions, given our assignment g of Gödel numbers to symbols, amounts to

$$\#s_0 \cdots s_n = code(\langle g(s_0), \dots, g(s_n) \rangle),$$

and assignment of super Gödel numbers to sequences of wffs to

$$\mathcal{G}(\langle \varphi_0, \dots, \varphi_n \rangle) = code(\langle \#\varphi_0, \dots, \#\varphi_n \rangle).$$

Fundamentals (cont.)

Recall also that we had the “decoding” function $exf : \mathbb{N}^2 \rightarrow \mathbb{N}$ s.t.

$$exf(\text{code}(\langle k_0, \dots, k_i, \dots, k_n \rangle), i) = k_i + 1$$

and a function $len : \mathbb{N} \rightarrow \mathbb{N}$ s.t.

$$len(\text{code}(\langle k_0, \dots, k_i, \dots, k_n \rangle)) = n + 1.^2$$

We will also need a concatenation function $\star : \mathbb{N}^2 \rightarrow \mathbb{N}$ so that if we have two expressions ϵ and ϵ' , we can compute the Gödel number of ϵ concatenated with ϵ' in terms of the Gödel numbers of ϵ and ϵ' respectively:

$$a \star b = a \cdot \prod_{j < len(b)} \pi_{j+len(a)}^{exf(b,j)}$$

²These functions are only partial since 0 is not a code number and 1 by courtesy is only the code number of the empty sequence. If you like, you can make them total by setting $exf(0, i) = exf(1, i) = len(0) = len(1) \equiv 1$.

Fundamentals (cont.)

Note that

$$a \star (b \star c) = (a \star b) \star c,$$

and that \star is primitive recursive, since the generalized product in its definition can be defined in terms of the bounded minimization of a p.r. property:

$$\prod_{j < \text{len}(b)} \pi_{j + \text{len}(a)}^{\text{exf}(b, j)} =$$

$$(\mu x \leq \pi_{a+b}^{a+b})(\forall j < \text{len}(b))[\text{exf}(x, j + \text{len}(a)) = \text{exf}(b, j)].$$

Before showing that the set of Gödel numbers of terms is p.r. we more modestly show this for variables and numerals.

Variables as Expressions

We showed awhile ago that the set of even numbers is p.r., and this is just the set of g numbers of variables. Obviously,

$$\#v_n = 2^{g(v_n)+1} = 2^{2n+1}.$$

So, the set of Gödel numbers of *expressions* that are sequences of length 1 consisting of a single variable is

$$\begin{aligned} \text{Vex} &:= \{\#e \mid e \text{ is a variable}\} \\ &= \{2^{2n+1} \mid n \in \mathbb{N}\}. \end{aligned}$$

Let $\text{Var}(v_0)$ be the wff

$$(\exists x \leq v_0)(v_0 = 2^{2x+1}).$$

$\text{Var}(v_0)$ defines Vex in \mathfrak{N} and represents it in \mathbb{Q} since Vex is primitive recursive.

Numerals

We can also define by primitive recursion a function $num : \mathbb{N} \rightarrow \mathbb{N}$ that takes us from a number to the Gödel number of its numeral:

$$\begin{aligned} num(0) &= 2^{14} \\ num(S(n)) &= 2^{16} \star num(n). \end{aligned}$$

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Keep in mind that num is a function in the metalanguage, not a defined function symbol in the object language (In function space parlance, $num \in \mathbb{N}^{\mathbb{N}}$.) The function num is useful in expressing the Gödel numbers of object language expressions, as we will shortly see.

diag is P.R.

Recall the diagonalization of $\alpha(v_0)$ is

$$\exists v_0 (v_0 = \ulcorner \alpha(v_0) \urcorner \wedge \alpha(v_0)).$$

Translated into primitives, this is really the wff

$$\neg \forall v_0 (v_0 = \ulcorner \alpha(v_0) \urcorner \rightarrow \neg \alpha(v_0)).$$

Suppose $\# \alpha(v_0) = n$. The Gödel no. $diag(n)$ of the diagonal is thus

$$\# \neg \forall v_0 (v_0 = \star num(n) \star \# \rightarrow \neg \star n \star \#),$$

i.e.,

$$(2^6 \cdot 3^{10} \cdot 5^1 \cdot 7^4 \cdot 11^1 \cdot 13^{12}) \star num(n) \star (2^8 \cdot 3^6) \star n \star 2^2,$$

which is patently p.r. given that \star is p.r. ■

Term Sequences

In order to find a wff that represents the set of Gödel nos. of terms, we'll first define the set of super Gödel nos. of construction sequences for terms, where a construction sequence for terms is a sequence of expressions $\langle \epsilon_0, \dots, \epsilon_n \rangle$ s.t. for all $k \leq n$ either

1. ϵ_k is either **0** or a variable, or
2. there exist $i, j < k$ such that
 - 2.1 $\epsilon_k = \mathbf{S}\epsilon_i$,
 - 2.2 $\epsilon_k = +\epsilon_i\epsilon_j$, or
 - 2.3 $\epsilon_k = \times\epsilon_i\epsilon_j$.

If $\langle t_0, \dots, t_n \rangle$ is a construction sequence for terms, then

$$a = \pi_0^{\#t_0+1} \dots \pi_n^{\#t_n+1}$$

is its super Gödel number.

Term Sequences (cont.)

Let $Termseq(\bar{a})$ be true in \mathfrak{N} iff a is the super Gödel number of a construction sequence for terms. The obvious definition is:

$Termseq(v_0) =_{df}$

$$(\forall k < len(v_0)) \{ (exf(v_0, k) - 1 = 2^{14} \vee Var(exf(v_0, k) - 1) \vee$$

$$(\exists i < k)(exf(v_0, k) - 1 = 2^{16} \star (exf(v_0, i) - 1)) \vee$$

$$(\exists i, j < k)(exf(v_0, k) - 1 = 2^{18} \star (exf(v_0, i) - 1) \star (exf(v_0, j) - 1)) \vee$$

$$(\exists i, j < k)(exf(v_0, k) - 1 = 2^{20} \star (exf(v_0, i) - 1) \star (exf(v_0, j) - 1)) \}$$

This is built up from p.r. predicates by properties and bounded quantification and hence is p.r.

The Set of Gödel Nos. of Terms is P.R.

We can define the predicate $Term(\ulcorner t \urcorner)$ for the set of Gödel nos. of terms t as follows. A term is the last member of some construction sequence of terms. So,

$$Term(v_0) =_{df} (\exists x \leq B(v_0))(Termseq(x) \wedge v_0 + 1 = \text{exf}(x, \text{len}(x) - 1)),$$

will do if we can formulate an acceptable bound $B(v_0)$ for x . If $\#t = n$ and $\text{len}(n) = \ell + 1$, then t will have a construction sequence a of length no more than $\ell + 1$. Let it be $\langle t_0, t_1, \dots, t_\ell \rangle$. Then

$$a := \mathcal{G}(\langle t_0, \dots, t_\ell \rangle) = \pi_0^{\#t_0+1} \cdot \pi_1^{\#t_1+1} \dots \pi_\ell^{\#t_\ell+1}.$$

Clearly, for each i , $\#t_i \leq n$, so $a \leq (\pi_\ell^{n+1})^{\ell+1}$. So

$$B(v_0) =_{df} (\pi_{\text{len}(v_0)}^{v_0+1})^{\text{len}(v_0)}$$

should do. This establishes that $Term(v_0)$ is p.r.

The Set of Gödel Nos. of Wffs is P.R.

Atomic wffs are all equations, officially of the syntactic form $= t_1 t_2$, where t_1 and t_2 are terms. So the following defines the set of Gödel nos. of atomic wffs:

$$Atom(v_0) =_{df} (\exists x, y \leq v_0)(Term(x) \wedge Term(y) \wedge (v_0 = 2^{12} * x * y)),$$

which is clearly p.r.

The definition of $Wff(v_0)$ mimics that of $Term(v_0)$. One first defines a predicate $Formseq(v_0)$ for the super Gödel no. of a wff construction sequence along the lines of $Termseq(v_0)$. The details are left as an EXERCISE.

Bound Variables

Consider the three place relation: if variable x occurs at the $(i + 1)$ th position in φ , then that occurrence of x is bound. Call this $B(x, i, \varphi)$. We want to define the numerical relation

$$\{\langle \#x, i, \#\varphi \rangle \mid B(x, i, \varphi)\}.$$

Here is the defining wff.

$$\begin{aligned} \text{Bound}(v_0, v_1, v_2) =_{df} & \text{Var}(v_0) \wedge \text{Wff}(v_2) \wedge \\ & (\exists u, y, z < v_2)(v_2 = u \star 2^{10} \star v_0 \star y \star z) \wedge \\ & \text{Wff}(y) \wedge (\text{len}(u) + 1 < v_1 \wedge v_1 \leq \text{len}(u) + \text{len}(y) + 1) \end{aligned}$$

From the defining wff, it's clear that the relation is p.r.

Sentences (and Axioms)

We want to define the set of g.n.'s of sentences of \mathcal{L}_{PA} . A wff is a sentence iff every occurrence of a variable is bound. This suggests

$$\begin{aligned} \text{Sent}(v_0) =_{df} & \text{Wff}(v_0) \wedge \\ & (\forall i < \text{len}(v_0)) (\text{Even}(\text{exf}(v_0, i) - 1) \rightarrow \text{Bound}(2^{\text{exf}(v_0, i)}, i, v_0)). \end{aligned}$$

From here we want to go on to define the set of g.n.'s of logical axioms and of axioms of PA. The axioms of PA are entirely straightforward except for the induction schema, which takes some finesse.

Axioms (cont.)

Defining the set of g.n.'s of the logical axioms also takes some finesse. E.g., we need arithmetizations of the following notions:

- ▶ φ_t^x
- ▶ t is (legitimately) substitutable for x in φ
- ▶ α' is the result of replacing zero or more instances of t by t' in α (for atomic α)
- ▶ φ is a universal generalization of ψ

Proofs in PA

Defn. A proof sequence (derivation) d in PA is a sequence $\langle \varphi_0, \dots, \varphi_n \rangle$ s.t. for each $k \leq n$ either (i) φ_k is a logical axiom or an axiom of PA, or (ii) there exist $i, j < k$ s.t. $\langle \varphi_i, \varphi_j, \varphi_k \rangle \in MP$.

EXERCISE: Find a predicate $ModusPonens(v_0, v_1, v_2)$ that defines the relation $\{\langle \# \varphi, \# \psi, \# \chi \rangle \mid \langle \varphi, \psi, \chi \rangle \in MP\}$ and show that the relation is p.r.

EXERCISE: Find a predicate $ProofSeq(v_0)$ that defines the property $\{\mathcal{G}(d) \mid d \text{ is a proof sequence in PA}\}$ and show that the property is p.r.

EXERCISE: Find a predicate $Proof(v_0, v_1)$ that defines the property $\{\langle \mathcal{G}(d), \# \varphi \rangle \mid d \text{ is a proof in PA of } \varphi\}$ and show that the relation is p.r.