The First Incompleteness Theorem

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Mathematical Logic II

Spring 2022

ω-Completeness and ω-Consistency

Def. A numerical theory T is ω-**complete** iff for any wff $\varphi(x)$, whenever $T \vdash \varphi(\overline{n})$ for each $n \in \mathbb{N}$, we also have $T \vdash \forall x \varphi(x)$.

Def. T is ω-**incomplete** iff not ω-complete, i.e., for some wff $\varphi(x)$, $T \vdash \varphi(\overline{n})$ for each $n \in \mathbb{N}$, but $T \nvdash \forall x \varphi(z)$.

Def. T is ω-**consistent** iff for all wff $\varphi(x)$, if $T \vdash \varphi(\overline{n})$ for all $n \in \mathbb{N}$, then $T \nvdash \exists x \neg \varphi(x)$.

Def. T is ω-**inconsistent** iff not ω-consistent, i.e., for some wff $\varphi(x)$, $T \vdash \varphi(\overline{n})$ for each $n \in \mathbb{N}$, but $T \vdash \exists x \neg \varphi(x)$.

N.B. ω-consistency implies consistency but not vice-versa.

Construction of a Gödel Sentence G

Since $diag(n)$ is p.r., there is a predicate $Diag(v_0, v_1)$ that represents it in Q. Let

$$Gdl(v_0, v_1) =_{df} \exists z(Diag(v_1, z) \wedge Proof(v_0, z)).$$

Collapsing the distinction between sentences and their Gödel numbers and between sequences of sentences and their super Gödel numbers, $Gdl(v_0, v_1)$ says that v_0 is a proof in PA of some diagonalization of v_1.

But since the diagonalization of a wff is unique, $Gdl(v_0, v_1)$ says in effect,

v_0 is a proof in PA of the diagonalization of v_1.

Now let

$$U(v_0) =_{df} \neg\exists x \; Gdl(x, v_0).$$

So, $U(v_0)$ says that there is no proof in PA of the diagonalization of v_0.

Finally, let G be the diagonalization of $U(v_0)$, i.e.,

$$G =_{df} \exists y(y = \ulcorner U(v_0) \urcorner \; \wedge \; U(y)).$$

What does this say?

What G Says

We have straightforwardly:

$$\vdash G \leftrightarrow U(\ulcorner U(v_0) \urcorner).$$

Unpacking the definition of $U(v_0)$:

$$\vdash G \leftrightarrow \neg \exists x \; Gdl(x, \ulcorner U(v_0) \urcorner),$$

and, unpacking the definition of Gdl:

$$\vdash G \leftrightarrow \neg \exists x \exists z (Diag(\ulcorner U(v_0) \urcorner, z) \wedge \; Proof(x, z)).$$

Thus,

$$\vdash G \leftrightarrow \forall z (Diag(\ulcorner U(v_0) \urcorner, z) \rightarrow \neg \exists x \; Proof(x, z)).$$

What G Says (cont.)

Now in virtue of the way that G was defined,

$$diag(\#U(v_0)) = \#G.$$

So, since $Diag(v_0, v_1)$ represents *diag* in Q and hence functionally represents it in Q,

$$Q \vdash \forall z(Diag(\ulcorner U(v_0)\urcorner, z) \leftrightarrow z = \ulcorner G \urcorner).$$

Thus,

$$Q \vdash G \leftrightarrow \forall z(z = \ulcorner G \urcorner \rightarrow \neg \exists x\ Proof(x, z)),$$

and hence

$$Q \vdash G \leftrightarrow \neg \exists x\ Proof(x, \ulcorner G \urcorner).$$

So, assuming Q, G asserts (indirectly) that there is no proof of G.

An Easy Semantic Argument

Although Gödel's 1st incompleteness theorem does not proceed this way, we can now make an easy semantic argument for the incompleteness of PA. The idea of the standard model $\mathfrak{N}$ and truth in $\mathfrak{N}$ are post-Gödel notions (due to Tarski). The argument takes the form: *if* $\mathfrak{N}$ *is a model of PA, then PA is incomplete*.

More specifically: Suppose that $\mathfrak{N}$ is a model of PA and that $PA \vdash G$. Then there is a deduction d of G in PA, and $\models_{\mathfrak{N}} Proof(\mathcal{G}(d), \ulcorner G \urcorner)$. Thus, $\models_{\mathfrak{N}} \exists x\, Proof(x, \ulcorner G \urcorner)$, and so $\models_{\mathfrak{N}} \neg G$. But we also have $\models_{\mathfrak{N}} G$, since by assumption $\mathfrak{N} \in$ Mod PA and $PA \vdash G$. So if $\mathfrak{N} \in$ Mod PA, then $PA \nvdash G$.

An Easy Semantic Argument (cont.)

As for the negation of G, suppose that $PA \vdash \neg G$. Then, assuming that $\mathfrak{N} \in \text{Mod PA}$, $\models_{\mathfrak{N}} \exists x\, Proof(x, \ulcorner G \urcorner)$. So, for some particular $n \in \mathbb{N}$, $\models_{\mathfrak{N}} Proof(\overline{n}, \ulcorner G \urcorner)$. Since $Proof(v_0, v_1)$ defines in $\mathfrak{N}$ the relation

$$\{\langle n, m \rangle \mid n \text{ is the super g.n. of a PA-deduction of wff with g.n. } m\},$$

there is a PA-deduction d of G. I.e., $PA \vdash G$. But that means PA is inconsistent, and so $\mathfrak{N} \notin \text{Mod PA}$. Thus, $PA \nvdash \neg G$. ■

Proof Theory: Why PA does not prove G

Lemma. If PA is consistent, then $PA \nvdash G$.

Pf. Suppose $PA \vdash G$. Let $\mathcal{G}(d)$ be the super Gödel no. of some derivation d of G in PA. Since *Proof* represents in Q the deduction relation for PA, $Q \vdash Proof(\mathcal{G}(d), \ulcorner G \urcorner)$, and hence $PA \vdash Proof(\mathcal{G}(d), \ulcorner G \urcorner)$. But

$$PA \vdash G \leftrightarrow \neg \exists x \; Proof(x, \ulcorner G \urcorner).$$

So, on the assumption that $PA \vdash G$,

$$PA \vdash \forall x \neg Proof(x, \ulcorner G \urcorner),$$

and thus $PA \vdash \neg Proof(\mathcal{G}(d), \ulcorner G \urcorner)$, rendering PA inconsistent. ■

Proof Theory: Why PA, if ω-consistent, does not prove $\neg G$

Lemma. If PA is ω-consistent, then $PA \nvdash \neg G$.

Pf. Suppose PA is ω-consistent but $PA \vdash \neg G$. Since ω-consistency entails consistency, $PA \nvdash G$. Hence, for every $n \in \mathbb{N}$, $PA \vdash \neg Proof(\overline{n}, \ulcorner G \urcorner)$. By ω-consistency

$$PA \nvdash \exists x \neg\neg Proof(x, \ulcorner G \urcorner).$$

But

$$\begin{aligned} PA \nvdash \exists x \neg\neg Proof(x, \ulcorner G \urcorner) \quad &\Rightarrow \quad PA \nvdash \exists x\ Proof(x, \ulcorner G \urcorner) \\ &\Rightarrow \quad PA \nvdash \neg\neg\exists x\ Proof(x, \ulcorner G \urcorner) \\ &\Rightarrow \quad PA \nvdash \neg G \qquad \blacksquare \end{aligned}$$

The First Incompleteness Theorem

Theorem. Let T be any theory at least as strong as PA such that T is axiomatizable by a set of sentences, the set of Gödel nos. of which is p.r.[1] Then there is a Π_1 sentence (sentence of Goldbach type) G_T such that (i) if T is consistent, then $T \nvdash G_T$, and (ii) if T is ω-consistent, then $T \nvdash \neg G_T$.

Pf. Grant the supposition. Following the same procedure, we can show that the relation

$$\{\langle \mathcal{G}(d), \#\varphi \rangle \mid d \text{ is a derivation of } \varphi \text{ in } T\}$$

is p.r., and hence representable in T by a predicate $Proof_T(v_0, v_1)$ out of which we can construct a Gödel sentence G_T by diagonalization with the same properties, *mutatis mutandis*. ■

[1] T may also be in a language that extends $\mathcal{L}_{PA}$.

Comments on ω-Inconsistency and ω-Incompleteness

- Recall that Q is patently ω-incomplete (assuming Q is consistent).
- If PA is consistent, then PA is also ω-incomplete. (For then $PA \nvdash \forall x \neg\, Proof(x, \ulcorner G \urcorner)$ even though $PA \vdash \neg\, Proof(\overline{n}, \ulcorner G \urcorner)$ for every $n \in \mathbb{N}$.)
- If $\mathfrak{N} \in$ Mod PA, then PA is ω-consistent.
- If PA is consistent, there are arithmetic theories that are consistent but not ω-consistent. (One example is PA$+\neg G$. A simpler example: let $\varphi(v_0)$ be the predicate that asserts that if v_0 is a prime greater than 2, then v_0 is odd. Let the axioms of the theory be the sentences $\varphi(\overline{n})$, for each $n \in \mathbb{N}$ together with the sentence $\neg\forall x \varphi(x)$.)