Further Results Related to Incompleteness

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Some Terminology

Notation: If Σ is a set of sentences, let

$$\#\Sigma =_{df} \{\#\tau \mid \tau \in \Sigma\}.$$

Def. Say that a theory T is "**nice**" iff (i) T is consistent, (ii) T extends Q,[1] and (iii) T is axiomatizable with a set of sentences Σ s.t. $\#\Sigma$ is p.r. (T is p.r. axiomatizable).

Def. For any "nice" theory T, let $Proof_T(v_0, v_1)$ be the wff constructed in analogy with $Proof(v_0, v_1)$ for PA s.t.

$$\models_{\mathfrak{N}} Proof_T(v_0, v_1)\ [\![n, m]\!]$$

iff n is the super g.n. of a proof in T of the wff with g.n. m.

[1] Keep in mind this means that T can be in a language that extends $\mathcal{L}_{PA}$

Some Terminology(cont.)

Scholium. If T is not axiomatizable, then it's not at all clear how to define a wff such as $Proof_T(v_0, v_1)$. If T is p.r. axiomatizable, then there is a wff $Proof_T(v_0, v_1)$ which represents in Q the relation: n is the super g.n. of a proof in T of the wff with g.n. m, and so it's clear that being p.r. axiomatizable is a sufficient condition for the relation to be definable in $\mathfrak{N}$. Thus, insisting that T be "nice" is more than we need suppose. However, if T extends Q, then $Proof_T(v_0, v_1)$ will represent the relation in T as well, allowing us to diagonalize and so forth.

Requiring that T be consistent is not at all essential.

Some Terminology(cont.)

Notation: Assuming T is p.r. axiomatizable, let $Prov_T(v_0)$ be the wff $\exists x\ Proof_T(x, v_0)$. Call this the **provability predicate for** T.

N.B. This *defines* (in $\mathfrak{N}$) the property:

$$\{n \in \mathbb{N} \mid n = \#\varphi \text{ for some wff } \varphi \text{ s.t. } T \vdash \varphi\}.$$

However, it does NOT follow that it *represents* that property in T (or in Q). In fact, we will show later that it is **not** representable in T if T is consistent.

Easy Results about Provability Predicates

Theorem. Suppose that T extends Q and that T is p.r. axiomatizable. Then, $T \vdash Prov_T(\ulcorner\varphi\urcorner)$ if $T \vdash \varphi$.

Pf. Suppose $T \vdash \varphi$. Then there exists a derivation d of φ in T. Let n be the super g.n. of the derivation. Then

$$T \vdash Proof_T(\overline{n}, \ulcorner\varphi\urcorner),$$

since T extends Q and thus $Proof_T(v_0, v_1)$ represents

$$\{\langle n, m\rangle \mid n \text{ is the super g.n. of a proof in } T \text{ of the wff with g.n. } m\}$$

in T. It follows immediately by existential generalization that

$$T \vdash \exists x\ Proof_T(x, \ulcorner\varphi\urcorner). \qquad \blacksquare$$

Easy Results about Provability Predicates (cont.)

Theorem. Suppose T is a p.r. axiomatic extension of Q that is ω-consistent. Then $T \vdash \varphi$ if $T \vdash Prov_T(\ulcorner \varphi \urcorner)$.

Pf. We prove the contrapositive. Suppose that $T \nvdash \varphi$. The wff $Proof_T(v_0, v_1)$ represents in Q, and hence in T, the relation:

$$\{\langle n, m \rangle \mid n \text{ is the super g.n. of a proof in } T \text{ of wff with g.n. } m\}.$$

Hence, since $T \nvdash \varphi$, it follows that, for all $n \in \mathbb{N}$,

$$T \vdash \neg Proof_T(\overline{n}, \ulcorner \varphi \urcorner).$$

Then, since T is ω-consistent, $T \nvdash \exists x \neg\neg Proof_T(x, \ulcorner \varphi \urcorner)$, which is equivalent to $T \nvdash Prov_T(\ulcorner \varphi \urcorner)$. ■

The Diagonalization Lemma

Lemma. Suppose that T extends Q and is p.r. axiomatizable. Let $\varphi(x)$ be any wff in the language of T with exactly the variable x free. Then there is a sentence γ s.t.

$$T \vdash \gamma \leftrightarrow \varphi(\ulcorner \gamma \urcorner).$$

Pf. Assuming the language of T is "reasonable," i.e., it's set of g numbers is p.r., then the diagonalization function $diag_T$ for the language of T is p.r., and hence, since T extends Q, there is a predicate $Diag_T(v_0, v_1)$ that represents $diag_T$ in T. Let

$$\psi(v_0) =_{df} \forall x(Diag_T(v_0, x) \rightarrow \varphi(x)).$$

For any wff α, $\psi(\ulcorner \alpha \urcorner)$ asserts that the g.n of the diagonalization of α has the property expressed by φ.

The Diagonalization Lemma (cont.)

Now let γ be the diagonalization of $\psi(v_0)$. I.e.,

$$\gamma =_{df} \exists y(y = \ulcorner \psi(v_0) \urcorner \wedge \psi(y)).$$

Since $\vdash \gamma \leftrightarrow \psi(\ulcorner \psi(v_0) \urcorner)$, we have, unpacking ψ,

$$\vdash \gamma \leftrightarrow \forall x(Diag_T(\ulcorner \psi(v_0) \urcorner, x) \rightarrow \varphi(x)).$$

In other words, γ is equivalent to the assertion that the g.n. of the diagonalization of $\psi(v_0)$ has the property expressed by φ. And γ just *is* the diagonalization of $\psi(v_0)$. So, of course the g.n. of γ has the property expressed by φ. (Exclamation point!)

The Diagonalization Lemma (cont.)

In symbols rather than words, the argument goes like this. Since $Diag_T(v_0, v_1)$ *functionally* represents $diag_T$ in T,

$$T \vdash \forall x(Diag_T(\ulcorner \psi(v_0) \urcorner, x) \leftrightarrow x = \ulcorner \gamma \urcorner).$$

This, together with the equivalence

$$\vdash \gamma \leftrightarrow \forall x(Diag_T(\ulcorner \psi(v_0) \urcorner, x) \rightarrow \varphi(x)),$$

yields

$$T \vdash \gamma \leftrightarrow \forall x(x = \ulcorner \gamma \urcorner \rightarrow \varphi(x)),$$

and thus,

$$T \vdash \gamma \leftrightarrow \varphi(\ulcorner \gamma \urcorner). \quad \blacksquare$$

Incompleteness Again

Scholium. If we think of $\varphi(\ulcorner\cdot\urcorner)$ as a function that takes us from wffs to (equivalence classes of) wffs, then we can think of (the equivalence class of) γ as a fixed point of $\varphi(\ulcorner\cdot\urcorner)$. ■

We have shown, for any p.r. axiomatizable extension T of Q,

C1: if $T \vdash \sigma$, then $T \vdash Prov_T(\ulcorner\sigma\urcorner)$, and

Cω: if T is ω-consistent and $T \vdash Prov_T(\ulcorner\sigma\urcorner)$, then $T \vdash \sigma$.

It then follows immediately from the diagonalization lemma that:

Theorem. Suppose T is a p.r. axiomatizable extension of Q. Then there is a sentence γ s.t. (i) if T is consistent, then $T \nvdash \gamma$, and (ii) if T is ω-consistent, then $T \nvdash \neg\gamma$.

Incompleteness Again (cont.)

Pf. Take γ to be a fixed point of $\neg Prov_T(v_0)$, so we have

$$(*)\quad T \vdash \gamma \leftrightarrow \neg Prov_T(\ulcorner \gamma \urcorner).$$

(i) Suppose T is consistent but $T \vdash \gamma$. Then by C1 above, we have that $T \vdash Prov_T(\ulcorner \gamma \urcorner)$. From $(*)$, we then get that $T \vdash \neg\gamma$, so T is inconsistent.

(ii) Suppose T is ω-consistent but $T \vdash \neg\gamma$. Then, by $(*)$, $T \vdash Prov_T(\ulcorner \gamma \urcorner)$. From C$\omega$ it then follows that $T \vdash \gamma$, so T isn't even consistent, much less ω-consistent. ∎

Provability Is Unrepresentable

Theorem. Suppose T is a consistent, p.r. axiomatizable extension of Q (i.e., T is "nice"). Then there is no wff that represents in T the set:

$$\{n \in \mathbb{N} \mid n \text{ is the g.n. of a wff provable from } T\}.$$

Pf. Suppose that $\rho(v_0)$ represents the set. That entails that for any sentence σ:
(i) if $T \vdash \sigma$, then $T \vdash \rho(\ulcorner\sigma\urcorner)$, and
(ii) if $T \nvdash \sigma$, then $T \vdash \neg\rho(\ulcorner\sigma\urcorner)$.
By the diagonalization lemma, there is a wff γ s.t.

$$(*) \quad T \vdash \gamma \leftrightarrow \neg\rho(\ulcorner\gamma\urcorner).$$

If $T \nvdash \gamma$, then, by (ii), $T \vdash \neg\rho(\ulcorner\gamma\urcorner)$. But, by $(*)$, then $T \vdash \gamma$.

Provability Is Unrepresentable (cont.)

On the other hand, if $T \vdash \gamma$, then, by (i), $T \vdash \rho(\ulcorner\gamma\urcorner)$. But, by $(*)$, then $T \vdash \neg\gamma$, rendering T inconsistent. ■

Cor. If T is a consisent, p.r. axiomatizable extension of Q (i.e., if T is "nice"), then the set

$$\{n \in \mathbb{N} \mid n \text{ is the g.n. of a wff provable from } T\}$$

is not p.r.

Pf. If it were p.r., then it would be representable in T. ■

The Rosser Provability Predicate

In what follows, fix T to be a p.r. axiomatizable extension of Q and, dropping subscripts, let $Proof(v_0, v_1)$ be its provability predicate. Let $\overline{Proof}(v_0, v_1)$ represent in T the relation

$$\{\langle \mathcal{G}(d), \#\varphi \rangle \mid d \text{ is a proof in } T \text{ of } \neg\varphi\}.$$

Let

$$RProv(v_0) =_{df} \exists x(Proof(x, v_0) \wedge (\forall y \leq x)\neg\overline{Proof}(y, v_0)).$$

Thus, $RProv(v_0)$ says that v_0 is the g.n. of a wff for which there is a proof in T and there is no "smaller" proof in T of the negation of that wff. This is called the **Rosser provability predicate**.

The Gödel-Rosser Theorem

Theorem. If T is consistent, then T is incomplete.

Pf. By the diagonalization lemma, there is a fixed point γ of the negation of the Rosser provability predicate, i.e., there exists a wff γ s.t.

$$T \vdash \gamma \leftrightarrow \neg RProv(\ulcorner \gamma \urcorner).$$

So, γ asserts (indirectly) that γ is not Rosser-provable, i.e., γ is not provable by a sequence of wffs for which there is no "smaller" sequence proving $\neg\gamma$. Now suppose that T is consistent. We claim that (i) $T \nvdash \gamma$ and (ii) $T \nvdash \neg\gamma$.

Proof that $T \nvdash \gamma$

(i) Suppose that $T \vdash \gamma$. Then $T \vdash \neg RProv(\ulcorner\gamma\urcorner)$, i.e.,

$$T \vdash \neg\exists x(Proof(x, \ulcorner\gamma\urcorner) \wedge (\forall y \leq x)\neg\overline{Proof}(y, \ulcorner\gamma\urcorner)).$$

Thus

$$T \vdash \forall x(\neg Proof(x, \ulcorner\gamma\urcorner) \vee (\exists y \leq x)\overline{Proof}(y, \ulcorner\gamma\urcorner)).$$

Consider any $n \in \mathbb{N}$. Suppose n is the super g.n. of a proof of γ. Then $T \vdash Proof(\overline{n}, \ulcorner\gamma\urcorner)$. Thus,

$$T \vdash (\exists y \leq \overline{n})\overline{Proof}(y, \ulcorner\gamma\urcorner).$$

Since T extends Q and Q is order-adequate,

$$T \vdash \overline{Proof}(\mathbf{0}, \ulcorner\gamma\urcorner) \vee \cdots \vee \overline{Proof}(\overline{n}, \ulcorner\gamma\urcorner).$$

Proof that $T \nvdash \gamma$ (cont.)

Since for any $m \in \mathbb{N}$, either

$$T \vdash \overline{Proof}(\overline{m}, \ulcorner\gamma\urcorner)$$

or

$$T \vdash \neg\overline{Proof}(\overline{m}, \ulcorner\gamma\urcorner),$$

it follows that for some $k \leq n$ that

$$T \vdash \overline{Proof}(\overline{k}, \ulcorner\gamma\urcorner).$$

Thus, $T \vdash \neg\gamma$ in contradiction to the assumption that T is consistent. ■

Proof that $T \nvdash \neg\gamma$

(ii) Suppose that $T \vdash \neg\gamma$. Then $T \vdash \overline{Proof}(\overline{m}, \ulcorner\gamma\urcorner)$ for some $m \in \mathbb{N}$. Since T is consistent, $T \vdash \neg Proof(\overline{n}, \ulcorner\gamma\urcorner)$ for all $n \in \mathbb{N}$. Thus,

$$T \vdash \neg Proof(\mathbf{0}, \ulcorner\gamma\urcorner) \wedge \cdots \wedge \neg Proof(\overline{m}, \ulcorner\gamma\urcorner).$$

Since Q, and hence T, is order-adequate,

$$T \vdash (\forall x \leq \overline{m})\neg Proof(x, \ulcorner\gamma\urcorner),$$

i.e.,

$$T \vdash \forall x(x \leq \overline{m} \rightarrow \neg Proof(x, \ulcorner\gamma\urcorner)).$$

Contrapositively,

$$(*) \quad T \vdash \forall x(Proof(x, \ulcorner\gamma\urcorner) \rightarrow x \nleq \overline{m}).$$

Proof that $T \nvdash \neg\gamma$ (cont.)

OK. Since we've assumed that $T \vdash \neg\gamma$, we have also that $T \vdash RProv(\ulcorner\gamma\urcorner)$, i.e.,

$$T \vdash \exists x(Proof(x, \ulcorner\gamma\urcorner) \wedge (\forall y \leq x)\neg\overline{Proof}(y, \ulcorner\gamma\urcorner)).$$

Now we reason inside of T. Existentially instantiating on the above line with an arbitrary individual c, we have

$$(**) \quad Proof(c, \ulcorner\gamma\urcorner) \wedge (\forall y \leq c)\neg\overline{Proof}(y, \ulcorner\gamma\urcorner).$$

The first conjunct, together with $(*)$, entails that $c \nleq m$. Since T is order adequate,

$$T \vdash \forall x(x \leq \overline{m} \vee \overline{m} \leq x).$$

Proof that $T \nvdash \neg\gamma$ (cont.)

Thus, we can infer that $\overline{m} \leq c$. By the second conjunct of $(**)$ we get immediately that $\neg\overline{Proof}(\overline{m}, \ulcorner\gamma\urcorner)$. So,

$$T \vdash \neg\overline{Proof}(\overline{m}, \ulcorner\gamma\urcorner),$$

Putting this together with the earlier result that

$$T \vdash \overline{Proof}(\overline{m}, \ulcorner\gamma\urcorner),$$

we find that T is inconsistent. ■

Craig's Theorem

Theorem. An effectively enumerable theory is axiomatizable.

Pf. Let Π be a procedure for effectively enumerating theory T as $\sigma_0, \sigma_1, \sigma_2, \ldots$, and let s_i be the number of steps of running Π that it takes to output σ_i. Revise Π so as to make sure a sentence of the form $\alpha \wedge \alpha \wedge \cdots \wedge \alpha$ is not outputted before α. For example, before the instruction to output a sentence σ, insert a GOTO instruction to the follow snippet of code:

- If σ is a conjunction of the form $\alpha \wedge \alpha \wedge \cdots \wedge \alpha$, first output α.
- RETURN

Call the revised procedure Π'.

Craig's Theorem (cont.)

Clearly $T = \text{Cn}\ \Sigma$ where Σ is the set of sentences:

$$\sigma_0, \sigma_1 \wedge \sigma_1, \sigma_2 \wedge \sigma_2 \wedge \sigma_2, \ldots.$$

Σ is decidable by the following procedure.

1. Given arbitrary sentence τ, τ has the form $\alpha \wedge \alpha \wedge \cdots \wedge \alpha$ for some number n of iterations of α (perhaps zero iterations).
2. Do $s_n + 2$ steps of procedure Π'.
3. If α has already been outputted, then $\tau \in \Sigma$. Otherwise $\tau \notin \Sigma$.

Thus, there is a decidable set Σ of sentences s.t. $T = \text{Cn}\ \Sigma$. ■

Craig's Theorem Once Again

Theorem. An effectively enumerable theory is axiomatizable.

Proof. Let $\sigma_0, \sigma_1, \sigma_2, \ldots$ be an effective enumeration of T. The axiomatization proceeds as follows. For each σ_n, perform Simplification ($\wedge$ Elim) on σ_n if each of the resulting conjuncts is tautologically equivalent to σ_n. Repeat until one obtains a conjunct κ that can no longer be further simplified into conjunct tautologically equivalent to κ (and hence to σ_n). Next conjoin κ with itself $n+1$ times (using right associativity, for the sake of definiteness). Call the resulting sentence σ'_n. Note that since κ is equivalent to σ_n, so is σ'_n. Let $\Sigma = \{\sigma'_n \mid n \in \mathbb{N}\}$ be the proposed axiomatization.

Proof of Craig's Theorem (cont.)

To show that Σ axiomatizes we need to show that (i) Cn $\Sigma = T$ and that (ii) Σ is decidable.

For (i), it suffices to note that each σ_n in the effective enumeration of T is tautologically equivalent to σ'_n.

For (ii), Σ is decidable according to the following effective procedure. Given an arbitrary sentence τ, determine the maximum number m such that τ has the form of a sentence conjoined (right associatively) with itself m times. Then $\tau \in \Sigma$ iff $\tau = \sigma'_m$. ■

Craig's Theorem Yet Another Time

Lemma. If $B \subset \mathbb{N}$ is a recursively enumerable set, then B is the range of a primitive recursive function.

Thesis. A set of natural numbers is recursively enumerable iff effectively enumerable.

Theorem (Craig). Let C be the closure of some recursively enumerable set $B \subseteq \mathbb{N}$ under some relation R, and let Q be a primitive recursive and symmetric subrelation of R such that for each $m \in B$, $\{n \mid \langle m, n \rangle \in Q\}$ is infinite. Then there exists a primitive recursive set A s.t. C is the closure of A under R.

Proof of the Original Theorem by Craig

Proof. Suppose f is a p.r. function that recursively enumerates B. Let

$$A = \{n \in \mathbb{N} \mid (\exists p \leq n) Q(f(p), n)\}.$$

Since A is gotten by bounded quantification from a p.r. function composed with a p.r. relation, A is p.r. Now, since for each $m \in B$ there exists an $n \in A$ s.t. $\langle m, n \rangle \in Q$ and since Q is symmetric, the closure of A under Q includes B. Since Q is a subrelation of R, the closure of A under R includes the closure of B under R, i.e., the closure of A under R includes C. And since Q is a subrelation of R, it follows that A is included in the closure of B under R, i.e., the closure of A under R is included in C. Therefore, C equals the closure of A under R. ■

Application of the Theorem

1. Let $\langle m, n \rangle \in R$ iff m is the super g.n. of a sequence s of sentences, n is the g.n. of a sentence σ, and $s \vdash \sigma$.
2. Let B be the set of g.n.'s of an effectively enumerable theory T, so that B is already closed under R.
3. Let $\langle m, n \rangle \in Q$ iff m and n are g.n.'s. of sentences σ and τ s.t. one is the iterated conjunction of the other some finite number of times.
4. Then A is the set of g.n.'s of a p.r. axiomatization of T.

Discussion of the Application

What, then, intuitively, is a decision procedure for determining, given an e.e. theory T whether a sentence σ is one of the resulting axioms?

1. Let $n = \#\sigma$.
2. Compute $f(0), f(1), \ldots, f(n)$.
3. Check to see if any $f(i)$ is the g.n. of a sentence τ such that σ is a conjunction of τ with itself a certain number of times, or vice-versa. If yes, then σ is an axiom, if not, then σ is not.

Why does this work?

Craig's Theorem (cont.)

Scholium. The procedure for determining whether an arbitrary sentence τ is a member of Σ did not involve any unbounded searches. Since it has been argued that procedures for computing functions involving only bounded searches indicate that the function is p.r., it follows that the characteristic function of the property of being the g.n. of a member of Σ is p.r. In other words, if T is effectively enumerable, then T is p.r. axiomatizable.

Cor*. Th $\mathfrak{N}$ is not effectively enumerable.

Pf. Otherwise let A be an axiomatization of Th $\mathfrak{N}$. Rerun the incompleteness arguments using A in place of Q and PA.

Tarski's Theorem

Theorem. $\#\text{Th } \mathfrak{N}$ is not definable in $\mathfrak{N}$.

Pf. Suppose that $\varphi(v_0)$ defines $\#\text{Th } \mathfrak{N}$. Now, for an arbitrary sentence γ, we have

$$\begin{aligned} \models_{\mathfrak{N}} \gamma \quad &\Leftrightarrow \quad \#\gamma \in \#\text{Th } \mathfrak{N} \\ &\Leftrightarrow \quad \models_{\mathfrak{N}} \varphi(\ulcorner \gamma \urcorner). \end{aligned}$$

Thus, for any sentence γ,

$$(*) \ \ \models_{\mathfrak{N}} \gamma \leftrightarrow \varphi(\ulcorner \gamma \urcorner).$$

Tarski's Theorem (cont.)

Let

$$\psi(v_0) =_{df} \forall x(Diag(v_0, x) \to \neg\varphi(x)),$$

which then defines the set of g.n.'s of wffs whose diagonalizations are "false." I.e.,

$$\models_{\mathfrak{N}} \psi(v_0)\ [\![n]\!] \text{ iff } diag(n) \notin \#\text{Th}\ \mathfrak{N}.$$

Now let γ be the diagonalization of $\psi(v_0)$.

Tarski's Theorem (cont.)

As a matter of logic,

$$\models \gamma \leftrightarrow \psi(\ulcorner \psi(v_0) \urcorner).$$

Unpacking $\psi(v_0)$,

$$\models \gamma \leftrightarrow \forall x(Diag(\ulcorner \psi(v_0) \urcorner, x) \rightarrow \neg \varphi(x)).$$

Since $Diag(v_0, v_1)$ defines $diag(n)$ in $\mathfrak{N}$,

$$\models_{\mathfrak{N}} \forall x(Diag(\ulcorner \psi(v_0) \urcorner, x) \leftrightarrow x = \ulcorner \gamma \urcorner).$$

Tarski's Theorem (cont.)

Hence,

$$\models_{\mathfrak{N}} \gamma \leftrightarrow \forall x(x = \ulcorner \gamma \urcorner \to \neg\varphi(x)).$$

Thus

$$\models_{\mathfrak{N}} \gamma \leftrightarrow \neg\varphi(\ulcorner \gamma \urcorner).$$

This contradicts (*) above. Therefore, $\#\text{Th}\ \mathfrak{N}$ is not definable in $\mathfrak{N}$. ■